

ON SOME LOWER BOUNDS  
OF SOME SYMMETRY INTEGRALS

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We give LOWER BOUNDS of SYMMETRY INTEGRALS

$$I_f(N, h) \stackrel{\text{def}}{=} \int_N^{2N} \left| \sum_{|n-x| \leq h} \text{sgn}(n-x) f(n) \right|^2 dx$$

of arithmetic functions, ABBREV. A.F.,  $f : \mathbb{N} \rightarrow \mathbb{R}$ . (They will be REAL, now on).

This integral gives an idea of the ALMOST ALL (abbr. A.A.)  $f$  SYMMETRY in the SHORT INTERVAL (abbr., S.I.)  $[x-h, x+h]$ ; hereon  $h = o(x)$ , ACTUALLY  $h = [N^\theta]$ , where we

CALL  $\theta \in ]0, 1[$  the “WIDTH” (not the length,  $h$ ) of the S.I.

Our lower bounds of  $I_f$  IMPLY LOWER BOUNDS OF  $J_f$ , SAY,

$$J_f(N, h) \stackrel{\text{def}}{=} \int_N^{2N} \left| \sum_{x < n \leq x+h} f(n) - M_f(x, h) \right|^2 dx,$$

the SELBERG INTEGRAL, of the A.F.  $f : \mathbb{N} \rightarrow \mathbb{R}$ , where the EXPECTED MEAN-VALUE of the short sum  $\sum_{x < n \leq x+h} f(n)$  is

$$M_f(x, h) \approx h \left( \frac{1}{x} \sum_{n \leq x} f(n) \right) = \frac{h}{x} \sum_{d \leq x} g(d) \left[ \frac{x}{d} \right]$$

now on  $g := f * \mu$  (MÖBIUS FUNCTION), I.E.  $f(n) = \sum_{d|n} g(d)$ .

Taking SUPPORT of  $g$  INSIDE  $[1, Q]$ ,  $Q$  “small” w.r.t.  $x$

$$M_f(x, h) \approx h \sum_{d \leq x} \frac{g(d)}{d} = h \sum_{d \leq Q} \frac{g(d)}{d}.$$

Say,  $J_f$  HAS “MAIN TERM”  $M_f(x, h)$ , while  $I_f$  HAS NONE.

The CONNECTION BETWEEN  $I_f$  &  $J_f$  can be MADE EXPLICIT:

$$\begin{aligned}
I_f(N, h) &\ll J_f(N, h) + \int_N^{2N} |M_f(x, h) - M_f(x - h, h)|^2 dx + \\
&+ \int_N^{2N} \left| \sum_{x-h < n \leq x} f(n) - M_f(x - h, h) \right|^2 dx + \\
&+ \int_N^{2N} |f(x)|^2 + |f(x - h)|^2 dx
\end{aligned}$$

and we use the MODIFIED VINOGRADOV NOTATION (“ignores arbitrarily small powers”)

$$F(N, h) \lll G(N, h) \stackrel{def}{\iff} \forall \varepsilon > 0 \quad |F(N, h)| \ll_{\varepsilon} N^{\varepsilon} G(N, h),$$

to abbreviate  $\int_N^{2N} |f(x)|^2 + |f(x - h)|^2 dx \lll N$ , whenever

$$f \text{ is ESSENTIALLY BOUNDED } \stackrel{def}{\iff} \forall \varepsilon > 0 \quad |f(n)| \ll_{\varepsilon} n^{\varepsilon}$$

(abbrev.  $f \lll 1$ , esp. DIVISOR FUNCTION  $d(n) \lll 1$ )

LEAVING  $\lll N + h^3$  terms, NEGLIGIBLE, from the above

$$I_f(N, h) \ll J_f(N, h) + \int_N^{2N} |M_f(x, h) - M_f(x - h, h)|^2 dx$$

whence  $J_f(N, h) \gg I_f(N, h)$ , whenever  $M_f$  DEPENDS WEAKLY ON  $x$ , here.

HENCE (in good hypotheses on  $f \lll 1$ , REAL) LOWER B.DS OF  $I_f$  IMPLY LOWER BOUNDS FOR  $J_f$ , here.

Now on FOCUS ON  $I_f$  (instead,  $J_f$  HAS more calc.s due to MAIN TERM: above we USED ONLY  $\frac{d}{dx}M_f(x, h)$  IS SMALL).

In particular, we'll treat the CASE  $f = d_k$  OF THE  $k$ -DIVISOR FUNCTION, with GENERATING DIRICHLET SERIES  $\zeta^k$  (with the RIEMANN  $\zeta$  FUNCTION).

HOWEVER, we first GIVE FAIRLY GENERAL RESULTS.

Introducing the MIXED SYMMETRY INTEGRAL OF  $f, f_1 : \mathbb{N} \rightarrow \mathbb{R}$

$$I_{f, f_1}(N, h) \stackrel{def}{=}$$

$$\int_N^{2N} \sum_{|n-x| \leq h} \operatorname{sgn}(n-x) f(n) \sum_{|m-x| \leq h} \operatorname{sgn}(m-x) f_1(m) dx$$

we have, EXPANDING THE SQUARE,

$$I_{f-f_1} = I_f - 2I_{f, f_1} + I_{f_1}$$

whence (RECALL  $I_{f-f_1}(N, h) \geq 0$ )

$$(LB) \quad I_f(N, h) \geq 2I_{f, f_1}(N, h) - I_{f_1}(N, h).$$

This (LB) is true for ANY COUPLE OF REAL A.F. and GIVES A NON-TRIVIAL LOWER BD, PROVIDED  $I_{f_1}$  IS "SMALL".

Here  $f_1$  is an "AUXILIARY FUNCTION" (REAL,  $\lll 1$ ).

The MAIN POINT, here, KEEPING IN MIND APPLICATION TO

$$f = d_k \quad \& \quad \text{AUXILIARY } f_1 := g_1 * \mathbf{1}, \text{ I.E. } f_1(n) = \sum_{\substack{d|n \\ d \leq D}} 1$$

( $f_1$  is a kind of “RESTRICTED DIVISOR FUNCTION”, here)

is to DEFINE ( $L := \log N$ , with  $N \rightarrow \infty$  our MAIN VARIABLE)

a) the LEVEL,  $\lambda := \frac{\log Q}{L}$ , where  $f = g * \mathbf{1}$ ,  $\text{supp}(g) \subset [1, Q]$

b) the WIDTH,  $\theta := \frac{\log h}{L}$  (see above),  $0 < \theta < \frac{1}{2}$  NOW ON

c) the “AUXILIARY LEVEL”  $\delta := \frac{\log D}{L}$ , SMALL, see above  $g_1, f_1$

Introducing  $\delta$  SMALL is to APPLY (a kind of) LARGE SIEVE inequality (ACTUALLY, an elementary Lemma, proved through CAUCHY inequality FOR WELL-SPACED FAREY FRACTIONS).

In this way, we get (in suitable hypotheses for  $f$  &  $f_1$ )

$$I_{f, f_1}(N, h) \gg NhL^c, \quad c > 0 \text{ suitable}$$

and, from this and  $I_{f_1}$  “SMALL” (SAY, ESP.,  $I_{f_1}(N, h) \ll NL^3$ ,  $c > 3$ ), the (LB) inequality above gives the REQUIRED LOWER BOUND for  $I_f$  (WHENCE, for  $J_f$ ).

Before to proceed exposing our results, we'll give SOME MOTIVATION for our lower bounds study.

First of all, study of the “CLASSICAL” SELBERG INTEGRAL, SAY  $J = J_\Lambda$  :

$$J(N, h) = J_\Lambda(N, h) := \int_N^{2N} \left| \sum_{x < n \leq x+h} \Lambda(n) - h \right|^2 dx$$

(SEE THAT HERE  $M_f = M_\Lambda$  DOESN'T DEPEND AT ALL ON  $x$ !)

$$\text{VON-MANGOLDT } \Lambda(n) := \begin{cases} \log p & n = p^k, p \text{ PRIME}, k \geq 1 \\ 0 & \text{OTHERWISE} \end{cases}$$

is “WIDE-SPREADED” IN THE LITERATURE.

Less known (MUCH LESS!) is  $I_\Lambda$  the SYMMETRY INTEGRAL OF THE PRIMES:

$$I(N, h) = I_\Lambda(N, h) := \int_N^{2N} \left| \sum_{|n-x| \leq h} \Lambda(n) \text{sgn}(n-x) \right|^2 dx$$

appearing FIRST TIME in KACZOROWSKI-PERELLI work (90s). They give (see above) A LINK  $J_\Lambda \leftrightarrow I_\Lambda$  ( $\gg$  EASY,  $\ll$  HARD).

As the case  $f = \Lambda$  is hopeless to study (apart from conditional results), then, I (& SALERNO) started to study MORE GENERAL  $I_f$  (its asymptotic,  $f = \text{DIVISOR FUNCTION}$ , etc.).

Also, I have given upper bounds (WITH IWANIEC, ESP., treating Hecke eigenvalues  $f = \lambda$ ) for  $I_f$  for MANY A.F.  $\lll 1$ . (Turns out that, studying a fixed A.F.  $f : \mathbb{N} \rightarrow \mathbb{R}$ , into A.A.S.I., Perron formula lets  $I_f$  crop out of calculations ! For example, I & LAPORTA found this GENERAL PROPERTY, see Lemma 1 in Note Mat.).

We give our results : LOWER BOUNDS FOR MIXED SYMMETRY INTEGRALS.

(See that the METHOD MAY BE ADAPTED TO give ASYMPTOTIC FORMULÆ for them).

Then, if the auxiliary  $I_{f_1}$  is SMALL &  $M_f$ 's WEAKLY  $x$ -DEP., get LOWER BOUNDS (AS ABOVE) FOR  $J_f$ .

As usual,  $\|\alpha\| := \min_{n \in \mathbb{Z}} |\alpha - n|$  is the DISTANCE TO INTEGERS.

We STATE (& GIVE AN IDEA OF THE PROOF LATER) our LEMMA. Fix  $\varepsilon_0 > 0$  small (SAY,  $\varepsilon_0 < 10^{-9}$ ). Let  $N, h \in \mathbb{N}$ , with  $h \rightarrow \infty$  and  $h \ll N^{1/2-\varepsilon_0}$ . ASSUME  $g, g_1 : \mathbb{N} \rightarrow \mathbb{R}$  with  $1 \leq g, g_1 \ll 1$ . Then, on defining as above  $f := g * \mathbf{1}$ ,  $f_1 := g_1 * \mathbf{1}$  and  $I_{f, f_1}(N, h)$ , we have,  $\forall D, Q \geq 1, D \leq Q$ , such that  $DQ \ll N^{1-\varepsilon_0}$ , with  $\log D \geq \log h + \varepsilon_0 \log N$ ,

$$I_{f, f_1}(N, h) \geq 2CNh \sum_{2h < q \leq D} \frac{g(q)g_1(q)}{q} +$$

$$+ 2CN \left( \sum_{1 < \ell \leq D} \sum_{t|\ell} \frac{\mu(t)}{t^2} \left\| \frac{ht}{\ell} \right\| \sum_{k \leq \frac{D}{\ell}} \frac{g_1(\ell k)}{k} \sum_{\substack{n \leq \frac{D}{\ell} \\ n \neq k}} \frac{g(\ell n)}{n} \right),$$

where (when  $N \rightarrow \infty$ )  $C \in \mathbb{R}$  can be chosen as any  $0 < C < 1$ .

(The second term on right hand side is  $\geq 0$ , here.)

An EASY CALCULATION, then, lets us OBTAIN

THEOREM. *Fix WIDTH  $0 < \theta < 1/2$ , LEVEL  $0 < \lambda < 1$  and “AUXILIARY LEVEL”  $\delta > \theta$ , with  $\delta + \lambda < 1$ . Let  $N, h, D, Q \in \mathbb{N}$ , with  $h = [N^\theta]$ ,  $D = [N^\delta]$ ,  $Q = [N^\lambda]$ . ASSUME  $g, g_1$  and  $f, f_1$  DEFINED AS ABOVE. Then,  $\forall B = o(D/h)$ ,  $\frac{\log B}{\log N} \gg 1$ ,*

$$I_{f, f_1}(N, h) \geq 2CNh \sum_{2h < q \leq D} \frac{g(q)g_1(q)}{q} + 2CN \times$$

$$\times \left( \sum_{d \leq B} \frac{\mu(d)}{d^2} \sum_{1 < m \leq \frac{D}{d}} \left\| \frac{h}{m} \right\| \sum_{k \leq \frac{D}{md}} \frac{g_1(mdk)}{k} \sum_{\substack{n \leq \frac{D}{md} \\ n \neq k}} \frac{g(mdn)}{n} \right),$$

where (as  $N \rightarrow \infty$ ) we can choose any  $0 < C < 1$ .

(Again, second term on the right will be neglected, if  $f = d_k$ .)

As told before, CAN ADAPT TO an ASYMPTOTIC !

Here we will not give general lower bounds for  $I_f$  since we don't have (still) any hypotheses on  $I_{f_1}$  (auxiliary integral), to be able to apply (LB) with a non-trivial final lower bound for  $I_f$ .

(IN THE FUTURE, WE PLAN TO DO IT.)



Now, we give an idea of HOW TO APPLY IT FOR  $f = d_k$ .

From A KIND OF FLIPPING in  $d_k(n) = \sum_{q|n} d_{k-1}(q)$ :

$$d_k(n) = \sum_{j=0}^{k-1} \sum_{q|n} d_{k-1}^{(j)}(q),$$

where  $\forall j = 0, \dots, k-1$ , we DEFINE

$$d_{k-1}^{(j)}(q) := \sum_{\substack{d_1 \\ d_1 \cdots d_{k-1} = q \\ d_1, \dots, d_j < (N-h)^{1/k}}} \cdots \sum_{d_{k-1}} 1$$

and, CALLING

$$S_k^\pm(x) \stackrel{def}{=} \sum_{|n-x| \leq h} d_k(n) \operatorname{sgn}(n-x)$$

the SYMMETRY SUM OF  $f = d_k$ , WE GET

$$S_k^\pm(x) \sim \sum_{q \leq (N-h)^{1-\frac{1}{k}}} g(q) \chi_q(x)$$

(here “ $\sim$ ” stands FOR : LEAVE NEGLIGIBLE TERMS)

WHERE  $g(q) := \sum_{j=0}^{k-1} d_{k-1}^{(j)}(q)$ ,  $1 \leq g(q) \lll 1$  AND

$$\chi_q(x) := \sum_{\substack{|n-x| \leq h \\ n \equiv 0(q)}} \operatorname{sgn}(n-x).$$

(We abbreviate  $n \equiv a \pmod{q}$  with  $n \equiv a(q)$ , here).

Then, we may say:  $d_k$  has LEVEL  $\lambda = 1 - \frac{1}{k}$  (from flipping).

The MAIN POINT of our general Theorem is to APPLY the LARGE SIEVE to get

$$I_{f,f_1}(N, h) = \text{DIAGONAL TERMS} + \Delta,$$

with  $\Delta$  NON-DIAGONAL TERMS (“OFF THE DIAGONAL”).

(Here the DIAGONAL TERMS ARE OF THE KIND

$$Nh$$

APART FROM A FACTOR OF LOGARITHMS).

Then, in order to “SQUEEZE OUT” THE DIAGONAL,

$$\Delta \lll DQh = N^{\delta+\lambda}h$$

is smaller than  $Nh$ , SAY  $o(Nh)$ , PROVIDED  $\delta + \lambda < 1$ .

(For “TECHNICAL” REASONS,  $\delta < \theta$  HAS TO BE ASSUMED.)

Up to now, “GENERAL”  $f$ .

In application, since  $\lambda = 1 - \frac{1}{k}$ , CHOOSE  $\delta = \delta_k$  &  $\theta = \theta_k$  SUCH THAT :  $\theta_k < \delta_k < 1/k$ .

Then, we get the (RECALL  $L := \log N$ , here)

COROLLARY. Fix  $k \geq 5$  INTEGER. Let  $N, h \in \mathbb{N}$  give, say, WIDTH  $\theta = \theta_k$ ,  $0 < \theta_k < 1/k$ . Then

$$I_{d_k}(N, h) \gg_k NhL^{k-1}, \quad J_k(N, h) \gg_k NhL^{k-1}.$$

**Remark.** Here  $M_f = M_k : \frac{d}{dx}M_k(x, h)$  SMALL  $\Rightarrow J_k \gg I_{d_k}$ .

(EXPECTED,  $k \geq 3$ ; WE’LL IMPROVE OUR LOWER BOUNDS IN NEAR FUTURE.)

Here we write  $I_{d_k}$  to avoid confusion with  $2k$ -TH  $\zeta$  MOMENT,  
ON THE LINE:

$$I_k(T) \stackrel{def}{=} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt$$

which is STRICTLY ENTANGLED WITH  $J_k(N, h)$  (for certain ranges  $N = N(T)$ ,  $h = h(T)$ , here).

In fact, in a RECENT WORK  
(see Coppola at <http://arxiv.org/abs/0907.5561v1>)

$$J_k(N, h) \lll Nh \Rightarrow I_k(T) \lll T, \quad \forall k > 2$$

last bound the SO-CALLED (WEAK)  $2k$ -TH MOMENT PROBLEM.

(See that cases  $k = 1, 2$  are known, WITH GOOD ASYMPTOTICS: Ivić, Jutila, Motohashi, etc.).

Here, we are giving LOWER BOUNDS OF  $J_k$  (for  $k > 2$ ) OF THE SAME ORDER OF magnitude (with logs) of the DIAGONAL, MATCHING the REQUIRED UPPER BOUND to give NON-TRIVIAL UPPER BOUNDS FOR  $I_k(T)$ , there.

(Actually, see that the link wastes arbitrarily small powers.)

However, we are showing only ONE application of OUR FAIRLY GENERAL result.

We hope in the future, to cope with the case of other interesting REAL, ESSENTIALLY BOUNDED arithmetic functions.

WE GIVE OUR LEMMA PROOF, A SKETCH (Th.m's immediate).  
 PROOF. From ADDITIVE CHARACTERS ORTHOGONALITY

$$\chi_q(x) = \sum_{\substack{|r| \leq h \\ r \equiv -x(q)}} \text{sgn}(r) = \sum_{j < q} c_{j,q} e_q(jx)$$

with FINITE FOURIER COEFFICIENTS

$$c_{j,q} := \frac{1}{q} \sum_{|r| \leq h} \text{sgn}(r) e_q(rj)$$

SATISFYING

$$c_{dj',dq'} = \frac{1}{d} c_{j',q'}, \forall d, j', q' \in \mathbb{N},$$

WHENCE

$$\chi_q(x) = \sum_{\substack{\ell | q \\ \ell > 1}} \frac{\ell}{q} \sum_{j \leq \ell}^* c_{j,\ell} e_\ell(jx),$$

WITH

$$\sum_{j < q} |c_{j,q}|^2 = 2 \left\| \frac{h}{q} \right\|, \quad \sum_{j < \ell}^* |c_{j,\ell}|^2 = 2 \sum_{t|\ell} \frac{\mu(t)}{t^2} \left\| \frac{ht}{\ell} \right\|.$$

Then

$$I_{f,f_1} = \int_N^{2N} \sum_{q \leq Q} g(q) \chi_q(x) \sum_{d \leq D} g_1(d) \chi_d(x) dx =$$

$$\begin{aligned}
&= \sum_{q \leq D} g(q)g_1(q) \int_N^{2N} \chi_q^2(x) dx + \\
&+ \sum_{d \leq D} g_1(d) \sum_{\substack{q \leq Q \\ q \neq d}} g(q) \sum_{\substack{t|d \\ t > 1}} \frac{t}{d} \sum_{r \leq t}^* \overline{c_{r,t}} \sum_{\substack{\ell|q \\ \ell > 1}} \frac{\ell}{q} \sum_{j \leq \ell}^* c_{j,\ell} \int_N^{2N} e(\alpha x) dx
\end{aligned}$$

where  $\alpha := \frac{j}{\ell} - \frac{r}{t}$  gives  $0 \iff \ell = t \ \& \ j = r$  (DIAG.TERM)

$$(*) \quad \alpha \neq 0 \Rightarrow \int_N^{2N} e(\alpha x) dx \ll \frac{1}{\|\alpha\|}$$

AND

$$\|\alpha\| \geq \frac{1}{DQ}, \quad \forall \frac{j}{\ell} \neq \frac{r}{t}, \quad \forall t \leq D, \quad \forall \ell \leq Q$$

i.e., FAREY FRACTIONS WELL-SPACING PROPERTY.

ISOLATE DIAGONAL TERMS, RE-ARRANGE OFF-DIAGONAL,  
USE,  $\forall q > 2h$ ,

$$\int_N^{2N} \chi_q^2(x) dx = \int_N^{2N} \sum_{\frac{x-h}{q} \leq m \leq \frac{x+h}{q}} 1 dx = \frac{2Nh}{q} + \mathcal{O}(h),$$

INTO

$$\begin{aligned}
I_{f,f_1} &= \sum_{2h < q \leq D} g_1(q)g(q) \int_N^{2N} \chi_q^2(x) dx + \\
&+ N \sum_{1 < \ell \leq D} \sum_{k \leq \frac{D}{\ell}} \frac{g_1(\ell k)}{k} \sum_{\substack{n \leq \frac{D}{\ell} \\ n \neq k}} \frac{g(\ell n)}{n} \sum_{j \leq \ell}^* |c_{j,\ell}|^2 +
\end{aligned}$$

$$+ \sum_{q \leq 2h} g(q)g_1(q) \int_N^{2N} \chi_q^2(x)dx + \Delta, \text{ SAY,}$$

WITH NON-DIAGONAL TERMS  $\Delta$  BOUNDED AS

$$\lll \sum_{1 < t \leq D} \sum_{\substack{1 < \ell \leq Q \\ \ell \neq t}} \sum_{r \leq t}^* |c_{r,t}| \sum_{j \leq \ell}^* |c_{j,\ell}| \frac{1}{\left\| \frac{j}{\ell} - \frac{r}{t} \right\|}$$

APPLYING LEMMA 2 [C-Salerno, AA] to (\*)

$$\lll QD \sqrt{\sum_{1 < t \leq D} \left\| \frac{h}{t} \right\|} \sqrt{\sum_{1 < \ell \leq Q} \left\| \frac{h}{\ell} \right\|} \lll QDh = o(Nh),$$

and we used

$$\sum_{j \leq \ell}^* |c_{j,\ell}|^2 \leq \sum_{j < \ell} |c_{j,\ell}|^2 \ll \left\| \frac{h}{\ell} \right\|$$

TOGETHER WITH

$$\sum_{1 < \ell \leq Q} \left\| \frac{h}{\ell} \right\| = \sum_{1 < \ell \leq 2h} \mathcal{O}(1) + h \sum_{2h < \ell \leq Q} \frac{1}{\ell} \lll h. \quad \square$$

We explicitly remark that there is a kind of “waste”, in our lower bounds (skip  $\geq 0$  sums), as we are leaving “*many*” terms in our previous analysis.

However, this begins already with our Theorem and this, in turn, comes already from our Lemma.

In fact, we leave (see Lemma Proof) one sum over  $d \leq 2h$  for the sake of clarity (otherwise, the relative calculations are “cumbersome”).

Our Lemma, actually, (whence, our Theorem, too) has an “ASYMPTOTIC” VERSION (comprising the quoted sum), that allows us to improve our results, even towards an ASYMPTOTIC FORMULA for the mixed symmetry integrals.

We avoid this for the moment, also, because we are much more focused on the Selberg integral lower bounds (from the stated connection, even an asymptotic of the corresponding “pure” symmetry integral  $I_f$  doesn’t give asymptotics of  $J_f$ ).

(Actually, the real improvement comes from the terms of the Theorem we neglected, where the Möbius function renders more complicated our estimates.)

In passing, we note that the Russian school, see Linnik book on Dispersion Method, has found such kind of asymptotic formulæ, even for  $f = d_k, f_1 = d$  “mixed correlations”

$$\mathcal{C}_{f,f_1}(a) \stackrel{def}{=} \sum_{n \leq x} f(n)f(n+a), \quad \forall a \in \mathbb{N}.$$

(See that my work about these correlations allows to write asymptotics for the Selberg & the symmetry “mixed integrals”, using asymptotics for these correlations, compare my recent paper on [arxiv](#)).

It is a big hope, for our future work, to be able to give lower bounds to much more interesting integrals (maybe the Selberg classical one, too .).