- N O T A T I O N

We let $N \in \mathbf{N}$, $N \to \infty$ and (x is an integer) write $x \sim N$ for $N < x \le 2N$

$$S_f^{\pm}(x,h) := \sum_{|n-x|\le h} f(n)\mathrm{sgn}(n-x) \qquad \underline{\text{SYMMETRY SUM}} \text{ of } f$$

(f is a generic arithmetical function, usually real-valued)

$$I_f^{\pm}(x,h) := \sum_{x\sim N} \left|S_f^{\pm}(x,h)\right|^2 \qquad \underline{\text{SYMMETRY INTEGRAL}} \text{ of } f$$

(actually, we use this "discrete" version instead of the integral; like for :)

$$J_f^{\pm}(x,h) := \sum_{x\sim N} \left|\sum_{0<|n-x|\le h} f(n) - M_f(x,h)\right|^2 \quad \underline{\text{SELBERG INTEGRAL}} \text{ of } f$$

(actually, Selberg's integral is $\int_N^{2N} | \sum_{x<n\le x+h} \Lambda(n) - h|^2 dx$; here $\Lambda(p^k) = \log p$)

where $M_f(x,h)$ is the <u>MEAN VALUE</u> of f (esp. $f = \Lambda \Rightarrow M_f(x,h) = 2h$)

- M O T I V A T I O N

Our work starts from a Kaczorowski-Perelli paper RELATING I_f AND J_f IN THE CASE OF $f(n) = \Lambda(n)$.

The link is (in suitable ranges of h)

$$I_f \text{ "small"} \Leftrightarrow J_f \text{ "small"}$$

(in both cases, "small" means that "has a non-trivial bound").

Bounding I_Λ is hopeless at the moment.

Thus we studied "easier" arithmetical functions f (often REAL ones).

- S O M E   E S T I M A T E S   FOR $I_f(N,h)$

- $f(n) = d(n)$ divisor function C.-Salerno, AA(2004).
- $f(n) = \sum_{d|n} d^{-s}$ divisor sums ($\sigma > 0$) C., INTEGERS(2004).
- $f(n) = \mu^2(n)$ square-free numbers C., R.M.UN.PARMA(2004).
- $f(n) = \mu^2(n)g(n)$ sq.-f. supported f C., JIPAM(2004).
- $f(n) = \lambda(n)$ Hecke eigenvalues C.-Iwaniec, unpublished.

- T E C H N I Q U E S

Two kinds of methods :

$$\begin{cases} \left.\begin{array}{c} \text{“divisor sums”}(\sigma \geq 0) \\ \text{square-free numbers} \end{array}\right\} & = \text{LARGE SIEVE} \\ \text{Hecke eigenvalues} & = \text{DISPERSION \& ASYMPTOTICS of:} \\ & \quad \sum_{n\sim N} \lambda(n)\lambda(n+k) \text{ (available)} \end{cases}$$

(actually, here we need estimates for “short intervals” shifts k, i.e. $k \ll h$)

This last approach works also for the other cases, as

$$\sum_{n\sim N} d(n)d(n+k) \qquad \text{(BINARY ADDITIVE DIVISOR)}$$

$$\sum_{n\sim N} \mu^2(n)\mu^2(n+k) \qquad \text{(“BINARY ADDITIVE SQUAREFREE”)}$$

HAVE KNOWN ASYMPOTIC FORMULAE.

(NOT always KNOWN ASYMPTOTICS for $\sum_{n\sim N} f(n)f(n+k)$.

THEN, if possible, USE LARGE SIEVE.)

- R E C E N T   D E V E L O P M E N T

Since the Large-Sieve "BREAKS AT LEVEL 1/2", I.E.

THE MAGNITUDE OF d (DIVISORS) CAN'T GO BEYOND $\sqrt{N}$

we can introduce a new ingredient to circumvent the problem.

For another ("NEW") CLASS of functions we CAN USE L-S INEQUALITY.

This is

$$f(n) = \sum_{d|n} g(d) \quad , \quad \text{with } g(n) \ll \frac{1}{n^{\varepsilon}}, \ \varepsilon > 0.$$

In this case we use RAMANUJAN EXPANSION ($c_q(n)$=Ramanujan sum)

$$f(n) = \sum_{q=1}^{\infty} a_q(f) c_q(n),$$

which (thanks to WINTNER'S LEMMA & the bound for g) is ABSOLUTELY CONVERGENT $\forall n \in \mathbf{N}$, WITH $a_q(f)$ (RAMANUJAN COEFFICIENT)

$$a_q(f) = \sum_{\substack{m=1 \\ m \equiv 0 (\bmod q)}}^{\infty} \frac{g(m)}{m} = \frac{1}{q} \sum_{j \in \mathbf{N}} \frac{g(jq)}{j} \ll_{\varepsilon} \frac{1}{q^{1+\varepsilon}}.$$

This fact (splitting "LOW" and "HIGH" divisors) gives

L-S inequality $\Rightarrow$ $I_f(N,h)$ NON-TRIVIAL.

Thus we recover some previous cases, esp. $\sum_{d|n} d^{-s}$ (BUT WITH $\sigma > 0$)

Also, we get new ones, like the Jordan TOTIENT functions.

(The Large Sieve <u>*ALONE*</u> requires $g(n) \ll \frac{1}{n^{\varepsilon}}$ WITH $\varepsilon > 1/2$).

- D E T A I L S and I D E A S

In the sequel we will assume the following.

We will implicitly assume N, h to be natural numbers, with $h = h(N) \to \infty$ as $N \to \infty$ and $h = o(N)$.

(In this way we're studying our f in ALMOST ALL SHORT INTERVALS of the type $[x - h, x + h]$; here $N < x \le 2N$.)

In the following we'll write $L := \log N$.

• $f(n) = d(n)$ DIVISOR FUNCTION

For the divisor function $f(n) = d(n)$ we get both results for the symmetry integral I_d & for the Selberg integral J_d.

We quote here our main result about I_d in [C-Salerno,AA]:

$$h \le N^{1/2-\eta} \text{ for } \eta > 0 \Rightarrow I_d(N,h) = \frac{16}{\pi^2} N h \log^3 \frac{\sqrt{N}}{h} + \mathcal{O}(N h L^{5/2} \sqrt{\log L})$$

(See Corollary 1 and compare Theorem 1)

$$\text{MAIN INGREDIENTS} \begin{cases} \text{DIRICHLET'S HYPERBOLA ("FLIPPING")} \\ \text{"ASYMPTOTIC" LARGE SIEVE} \end{cases}$$

• $f(n)=\sum_{d|n} d^{-s} := \sigma_{-s}(n),$ say $(\sigma=\Re(s)>0)$ DIVISOR SUMS

Here the result is a little bit more involved. First of all, assume $\sigma>0$.

We need to define

$$\eta^{(h)}(s) := \sum_{n=1}^{\infty} \left\| \frac{h}{n} \right\| n^{-s}, \qquad s \in \mathbf{C}$$

where $\|\alpha\| := \min_{n\in\mathbf{Z}} |\alpha-n|$ is the DISTANCE TO THE INTEGERS.

In this case from Corollary 1 of [Coppola,INTEGERS] we get for the symmetry integral of f, say $I_s(N,h)$, the result

$$h=N^{\theta}, \theta < \frac{\min(\sigma,1/2)}{2+\sigma} \quad \Rightarrow \quad I_s(N,h) = 2\frac{|\zeta(1+s)|^2}{\zeta(2+2\sigma)} N\eta^{(h)}(2\sigma) + E(N,h),$$

with the remainder $E(N,h)=E(N,h,s)=o(N)$ (and here the o can depend however on s, σ, $\Im(s)$, $|s|$).

$$\text{MAIN INGREDIENTS} \begin{cases} \text{FLIPPING} \\ \text{ASYMPTOTIC LARGE SIEVE} \end{cases}$$

(Together with partial summation & standard estimates)

• $f(n) = \mu^2(n)$ SQUARE-FREE NUMBERS

From Corollary 2 of [Coppola,RIV.MAT.UNIV.PARMA] we get

$$h \le N^{4/15-\eta}, \text{for some } \eta > 0 \quad \Rightarrow \quad I_{\mu^2}(N,h) \ll N\sqrt{h}.$$

This time we use the well-known identity

$$\mu^2(n) = \sum_{d^2|n} \mu(d)$$

which allows us to "lower" the level (i.e., the magnitude of the divisors).

Also, our treatment doesn't look at the inner $\mu(d)$ (using only its boundedness), so may as well apply to any g like

$$g(n) = \sum_{d^2|n} b(d),$$

whatever is the (BOUNDED) $b(d)$.

In fact, we use:

MAIN INGREDIENTS:
- "LOW DENSITY" OF DIVISORS
- LARGE SIEVE INEQUALITY
- TREATMENT OF SPORADIC TERMS

- $f(n) = \mu^2(n) g(n)$ SQUARE-FREE SUPPORTED

We quote Thm.1.1 & Thm.1.3 [Coppola, JIPAM]. Write

$$\|g\| := \max_{N-h<n\le 2N+h} |g(n)|;$$

abbreviate: compl./c. = completely, mult. = multiplicative, add. = additive.

Assume $h/L^2 \to \infty$, as $N \to \infty$ and $J = J(N) \to \infty$, $J \ll \sqrt{h}/L$ as $N \to \infty$. Then

$$g \text{ compl.mult.} \Rightarrow I_g(N,h) \ll L^2 \max_{D \ll J} \sum_{d \sim D} d^2 I_{\mu^2 g}\left(\frac{N}{d^2}, \frac{h}{d^2}\right) + \frac{Nh^2}{J^2}\|g\|^2$$

$$g \text{ compl.mult.} \Rightarrow I_{\mu^2 g}(N,h) \ll L^2 \max_{D \ll J} \sum_{d \sim D} d^2 I_{g}\left(\frac{N}{d^2}, \frac{h}{d^2}\right) + \frac{Nh^2}{J^2}\|g\|^2$$

Also,

$$g \text{ c.add.} \Rightarrow I_g(N,h) \ll L^2 \max_{D \ll J} \sum_{d \sim D} d^2 I_{\mu^2 g}\left(\frac{N}{d^2}, \frac{h}{d^2}\right) + (\frac{Nh^2}{J^2} + NJ\sqrt{h}L^2)\|g\|^2$$

$$g \text{ c.add.} \Rightarrow I_{\mu^2 g}(N,h) \ll L^2 \max_{D \ll J} \sum_{d \sim D} d^2 I_{g}\left(\frac{N}{d^2}, \frac{h}{d^2}\right) + (\frac{Nh^2}{J^2} + NJL^2)\|g\|^2$$

At the same time, we also have, for the symmetry integral of $2^{\Omega(n)}$, with $\Omega(n)$ the total number of $p|n$ ($\forall \varepsilon > 0$ FIXED)

$$h = o\left(\frac{\sqrt{N}}{L}\right) \quad \Rightarrow \quad I_{2^{\Omega(n)}}(N,h) \ll Nh^{\frac{3}{2}} N^{\varepsilon}.$$

$$\text{MAIN INGREDIENTS} \begin{cases} \text{Link} \quad I_g \leftrightarrow I_{\mu^2 g} : \text{ELEMENTARY} \\ \text{Sym.Int.of } 2^{\Omega(n)} : \text{SYMMETRY of } d(n) \text{ in AP} \end{cases}$$

(AP = ARITHMETIC PROGRESSIONS)

• $f(n) = \lambda(n)$ HECKE EIGENVALUES

We need, here, to introduce a "SMOOTHED SYMMETRY INTEGRAL", i.e.

$$I_f^{(g)}(N,h) := \int_{-\infty}^{+\infty} \Big| \sum_{|n-x|\le h} f(n)g(n)\mathrm{sgn}(n-x) \Big|^2 dx,$$

(here g is "smooth" and $g(x) = 1$, $\forall x \in [N-h, 2N+h]$)

where, now, the symmetry integral is (again, for technical reasons) defined as

$$I_f(N,h) := \int_N^{2N} \Big| \sum_{|n-x|\le h} f(n)\mathrm{sgn}(n-x) \Big|^2 dx.$$

Using a POSITIVITY ARGUMENT, we get an upper bound for I_λ in terms of asymptotics for $I_\lambda^{(g)}$, i.e. its "smoothed" version.

These asymptotics, in turn, are obtained using the DISPERSION METHOD (which applies to symmetry integrals, even better than to Selberg integrals !), combined with ASYMPTOTICS FOR

$$\sum_{n\sim N} f(n)f(n+k) \qquad (k, \text{the shift, is } \mathcal{O}(h))$$

(here, say, $f(n) = \lambda(n)g(n)$, i.e. "SMOOTHED HECKE EIGENVALUES") AVAILABLE IN THE LITERATURE (due to Conrey-Iwaniec, see AA).

Summarizing (and avoiding technicalities) for the result:

$$h \ll N^{1/4} \ \Rightarrow \ I_\lambda(N,h) \ll_{k,q} NhL^3 \qquad (k = \text{weight}, q = \text{level})$$

and for the proof:

MAIN INGREDIENTS
- POSITIVITY
- DISPERSION
- ASYMPTOTICS FOR SMOOTHED E.V.

(E.V. = HECKE EIGENVALUES)