- NOTATION

We let \( N \in \mathbb{N}, N \to \infty \) and \((x\text{ is an integer})\) write \( x \sim N \) for \( N < x \leq 2N \)

\[
S_f^\pm(x, h) := \sum_{|n-x| \leq h} f(n) \text{sgn}(n-x) \quad \text{SYMMETRY SUM of } f
\]

\((f\text{ is a generic arithmetical function, usually real-valued)\)

\[
I_f^\pm(x, h) := \sum_{x \sim N} \left| S_f^\pm(x, h) \right|^2 \quad \text{SYMMETRY INTEGRAL of } f
\]

\(\text{(actually, we use this “discrete” version instead of the integral; like for :)\)

\[
J_f^\pm(x, h) := \sum_{x \sim N} \left| \sum_{0 < |n-x| \leq h} f(n) - M_f(x, h) \right|^2 \quad \text{SELBERG INTEGRAL of } f
\]

\(\text{(actually, Selberg’s integral is } \int_2^{2N} |\sum_{x<n \leq x+h} \Lambda(n) - h|^2 dx; \text{ here } \Lambda(p^k) = \log p)\)

where \(M_f(x, h)\) is the MEAN VALUE of \( f \) \((\text{esp. } f = \Lambda \Rightarrow M_f(x, h) = 2h)\)

- MOTIVATION

Our work starts from a Kaczorowski-Perelli paper RELATING \( I_f \) AND \( J_f \) IN THE CASE OF \( f(n) = \Lambda(n) \).

The link is (in suitable ranges of \( h \))

\[ I_f \text{ “small” } \iff J_f \text{ “small”} \]

\((\text{in both cases, “small” means that “has a non-trivial bound”)\)}

Bounding \( I_\Lambda \) is hopeless at the moment.

Thus we studied “easier” arithmetical functions \( f \) (often REAL ones).
- SOME ESTIMATES FOR $I_f(N, h)$

- $f(n) = d(n)$ divisor function C.-Salerno, AA(2004).

- $f(n) = \sum_{d|n} d^{-s}$ divisor sums $(\sigma > 0)$ C., INTEGERS(2004).

- $f(n) = \mu^2(n)$ square-free numbers C., R.M.UN.PARMA(2004).

- $f(n) = \mu^2(n)g(n)$ sq.-f. supported $f$ C., JIPAM(2004).

- $f(n) = \lambda(n)$ Hecke eigenvalues C.-Iwaniec, unpublished.

- TECHNIQUES

Two kinds of methods:

\[
\begin{align*}
\{ & \text{“divisor sums"} (\sigma \geq 0) \text{ square-free numbers} \\
& \text{Hecke eigenvalues} \} = \text{LARGE SIEVE} \\
& \text{DISPERSION} \& \text{ASYMPTOTICS of:} \\
& \sum_{n \sim N} \lambda(n)\lambda(n+k) \text{ (available)}
\end{align*}
\]

(Actually, here we need estimates for “short intervals” shifts $k$, i.e. $k \ll h$)

This last approach works also for the other cases, as

\[
\sum_{n \sim N} d(n)d(n+k) \quad \text{(BINARY ADDITIVE DIVISOR)}
\]

\[
\sum_{n \sim N} \mu^2(n)\mu^2(n+k) \quad \text{ (“BINARY ADDITIVE SQUAREFREE")}
\]

HAVE KNOWN ASYMPTOTIC FORMULAE.

\[
\left( \begin{array}{c}
\text{NOT always KNOWN ASYMPTOTICS for } \sum_{n \sim N} f(n)f(n+k) \text{.} \\
\text{THEN, if possible, USE LARGE SIEVE.}
\end{array} \right)
\]

2
- Recent Development

Since the Large-Sieve “breaks at level 1/2”, I.E.

The magnitude of \( d \) (divisors) can’t go beyond \( \sqrt{N} \)

we can introduce a new ingredient to circumvent the problem.

For another (“new”) class of functions we can use L-S Inequality.

This is

\[
    f(n) = \sum_{d|n} g(d) \quad \text{with } g(n) \ll \frac{1}{n^\varepsilon}, \ \varepsilon > 0.
\]

In this case we use Ramanujan expansion (\( c_q(n) \)=Ramanujan sum)

\[
    f(n) = \sum_{q=1}^{\infty} a_q(f) c_q(n),
\]

which (thanks to Wintner’s lemma & the bound for \( g \)) is absolutely convergent \( \forall n \in \mathbb{N} \), with \( a_q(f) \) (Ramanujan Coefficient)

\[
    a_q(f) = \sum_{\substack{m=1 \atop m \equiv 0 \pmod{q}}}^{\infty} \frac{g(m)}{m} = \frac{1}{q} \sum_{j \in \mathbb{N}} \frac{g(jq)}{j} \ll \varepsilon \frac{1}{q^{1+\varepsilon}}.
\]

This fact (splitting “low” and “high” divisors) gives

L-S inequality \( \Rightarrow I_f(N, h) \) non-trivial.

Thus we recover some previous cases, esp. \( \sum_{d|n} d^{-s} \) (but with \( \sigma > 0 \))

Also, we get new ones, like the Jordan Totient functions.

(The large sieve alone requires \( g(n) \ll \frac{1}{n^\varepsilon} \) with \( \varepsilon > 1/2 \)).
Details and Ideas

In the sequel we will assume the following.

We will implicitly assume \( N, h \) to be natural numbers, with \( h = h(N) \to \infty \) as \( N \to \infty \) and \( h = o(N) \).

(In this way we’re studying our \( f \) in ALMOST ALL SHORT INTERVALS of the type \([x-h, x+h]\); here \( N < x \leq 2N \).)

In the following we’ll write \( L := \log N \).

\( f(n) = d(n) \) DIVISOR FUNCTION

For the divisor function \( f(n) = d(n) \) we get both results for the symmetry integral \( I_d \) & for the Selberg integral \( J_d \).

We quote here our main result about \( I_d \) in [C-Salerno,AA]:

\[
h \leq N^{1/2-\eta} \text{ for } \eta > 0 \Rightarrow I_d(N,h) = \frac{16}{\pi^2} Nh \log^3 \frac{\sqrt{N}}{h} + O(NhL^{5/2} \sqrt{\log L})
\]

(See Corollary 1 and compare Theorem 1)

Main Ingredients

\{ DIRICHLET’S HYPERBOLA (“FLIPPING”) \}

\{ “ASYMPTOTIC” LARGE SIEVE \}
• \( f(n) = \sum_{d|n} d^{-s} := \sigma_{-s}(n) \), say \( \sigma = \Re(s) > 0 \)  

DIVISOR SUMS

Here the result is a little bit more involved. First of all, assume \( \sigma > 0 \).

We need to define

\[
\eta^{(h)}(s) := \sum_{n=1}^{\infty} \left\| \frac{h}{n} \right\| n^{-s}, \quad s \in \mathbb{C}
\]

where \( \| \alpha \| := \min_{n \in \mathbb{Z}} |\alpha - n| \) is the DISTANCE TO THE INTEGERS.

In this case from Corollary 1 of [Coppola, INTEGERS] we get for the symmetry integral of \( f \), say \( I_s(N, h) \), the result

\[
h = N^\theta, \theta < \frac{\min(\sigma, 1/2)}{2 + \sigma} \implies I_s(N, h) = 2 \frac{|\zeta(1 + s)|^2}{\zeta(2 + 2\sigma)} N \eta^{(h)}(2\sigma) + E(N, h),
\]

with the remainder \( E(N, h) = E(N, h, s) = o(N) \) (and here the \( o \) can depend however on \( s, \sigma, \Im(s), |s| \)).

MAIN INGREDIENTS \{ FLIPPING ASYMPTOTIC LARGE SIEVE \}

(Together with partial summation & standard estimates)
• $f(n) = \mu^2(n)$  SQUARE-FREE NUMBERS

From Corollary 2 of [Coppola,RIV.MAT.UNIV.PARMA] we get

$$h \leq N^{4/15 - \eta}, \text{for some } \eta > 0 \quad \Rightarrow \quad I_{\mu^2}(N,h) \ll N\sqrt{h}.$$  

This time we use the well-known identity

$$\mu^2(n) = \sum_{d^2|n} \mu(d)$$

which allows us to “lower” the level (i.e., the magnitude of the divisors).

Also, our treatment doesn’t look at the inner $\mu(d)$ (using only its boundedness), so may as well apply to any $g$ like

$$g(n) = \sum_{d^2|n} b(d),$$

whatever is the (BOUNDED) $b(d)$.

In fact, we use:

**MAIN INGREDIENTS**  

{“LOW DENSITY” OF DIVISORS

LARGE SIEVE INEQUALITY

TREATMENT OF SPORADIC TERMS


6
\[ f(n) = \mu^2(n)g(n) \quad \text{SQUARE-FREE SUPPORTED} \]

We quote Thm.1.1 & Thm.1.3 [Coppola, JIPAM]. Write

\[ \|g\| := \max_{N-h<n\leq 2N+h} |g(n)|; \]

abbreviate: compl./c. = completely, mult. = multiplicative, add. = additive.

Assume \( h/L^2 \to \infty \), as \( N \to \infty \) and \( J = J(N) \to \infty \), \( J \ll \sqrt{h}/L \) as \( N \to \infty \).

Then

\[ g \text{ compl.mult.} \Rightarrow I_g(N, h) \ll L^2 \max_{D \in J} \sum_{d \sim D} d^2 I_{\mu^2 g} \left( \frac{N}{d^2}, \frac{h}{d^2} \right) + \frac{Nh^2}{J^2} \|g\|^2 \]

\[ g \text{ compl.mult.} \Rightarrow I_{\mu^2 g}(N, h) \ll L^2 \max_{D \in J} \sum_{d \sim D} d^2 I_g \left( \frac{N}{d^2}, \frac{h}{d^2} \right) + \frac{Nh^2}{J^2} \|g\|^2 \]

Also,

\[ g \text{ c.add.} \Rightarrow I_g(N, h) \ll L^2 \max_{D \in J} \sum_{d \sim D} d^2 I_{\mu^2 g} \left( \frac{N}{d^2}, \frac{h}{d^2} \right) + \left( \frac{Nh^2}{J^2} + NJ\sqrt{h}L^2 \right) \|g\|^2 \]

\[ g \text{ c.add.} \Rightarrow I_{\mu^2 g}(N, h) \ll L^2 \max_{D \in J} \sum_{d \sim D} d^2 I_g \left( \frac{N}{d^2}, \frac{h}{d^2} \right) + \left( \frac{Nh^2}{J^2} + NJL^2 \right) \|g\|^2 \]

At the same time, we also have, for the symmetry integral of \( 2^{\Omega(n)} \), with \( \Omega(n) \) the total number of \( p|n \) (\( \forall \varepsilon > 0 \) FIXED)

\[ h = o \left( \frac{\sqrt{N}}{L} \right) \Rightarrow I_{2^{\Omega(n)}}(N, h) \ll Nh^{\frac{2}{3}}N^\varepsilon. \]

**MAIN INGREDIENTS**

\{ Link \( I_g \leftrightarrow I_{\mu^2 g} \): ELEMENTARY \Sym.Int.of 2^{\Omega(n)}: SYMMETRY of \( d(n) \) in AP \}

(AP = ARITHMETIC PROGRESSIONS)
• \( f(n) = \lambda(n) \) \hspace{1cm} \text{HECKE EIGENVALUES}

We need, here, to introduce a “SMOOTHED SYMMETRY INTEGRAL”, i.e.

\[
I_f^{(g)}(N, h) := \int_{-\infty}^{+\infty} \left| \sum_{|n-x| \leq h} f(n)g(n)\text{sgn}(n-x) \right|^2 dx,
\]

(here \( g \) is “smooth” and \( g(x) = 1, \forall x \in [N-h, 2N+h] \)

where, now, the symmetry integral is (again, for technical reasons) defined as

\[
I_f(N, h) := \int_{N}^{2N} \left| \sum_{|n-x| \leq h} f(n)\text{sgn}(n-x) \right|^2 dx.
\]

Using a POSITIVITY ARGUMENT, we get an upper bound for \( I_\lambda \) in terms of asymptotics for \( I_f^{(g)} \), i.e. its “smoothed” version.

These asymptotics, in turn, are obtained using the DISPERSION METHOD (which applies to symmetry integrals, even better than to Selberg integrals !), combined with ASYMPTOTICS FOR

\[
\sum_{n \sim N} f(n)f(n+k) \quad (k, \text{the shift, is } O(h))
\]

(here, say, \( f(n) = \lambda(n)g(n) \), i.e. “SMOOTHED HECKE EIGENVALUES”) AVAILABLE IN THE LITERATURE (due to Conrey-Iwaniec, see AA).

Summarizing (and avoiding technicalities) for the result:

\[
h \ll N^{1/4} \Rightarrow I_\lambda(N, h) \ll_{k,q} NhL^3 \quad (k = \text{weight}, q = \text{level})
\]

and for the proof:

\[
\text{MAIN INGREDIENTS} \left\{ \begin{array}{l}
\text{POSITIVITY} \\
\text{DISPERSION} \\
\text{ASYMPTOTICS FOR SMOOTHED E.V.}
\end{array} \right. 
\]

(E.V. = HECKE EIGENVALUES)