Introducing weighted Selberg integrals

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Atle Selberg, in 1946, studied prime numbers in almost all short intervals [S] estimating

$$\int_N^{2N} \Big| \sum_{x<n\le x+H} \Lambda(n) - H \Big|^2 \mathrm{d}x,$$

where Λ is the well-known von Mangoldt function, namely $\Lambda(p^r) \stackrel{def}{=} \log p$, $\forall p$ primes and $\forall r \in \mathbb{N}$, and 0 outside prime-powers. This is the "classical", so to speak, Selberg integral.

In the 1990s, Kaczorowski & Perelli [KP] discovered a remarkable link between classic Selberg integral above and the "symmetry integral of the primes",

$$\int_N^{2N} \Big| \sum_{x<n\le x+H} \Lambda(n) - \sum_{x-H\le n<x} \Lambda(n) \Big|^2 \mathrm{d}x.$$

Discrete versions of these integrals are, for any arithmetic function f, the Selberg integral of f,

$$J_f(N,H) \stackrel{def}{=} \sum_{N<x\le 2N} \Big| \sum_{x<n\le x+H} f(n) - M_f(x,H) \Big|^2,$$

$M_f(x,H)$ is the "short interval mean-value" (expected value of short sum) and the symmetry integral of f,

$$J_{\mathrm{sgn},f}(N,H) \stackrel{def}{=} \sum_{N<x\le 2N} \Big| \sum_{x-H\le n\le x+H} \mathrm{sgn}(n-x) f(n) \Big|^2.$$

I introduced(2010) the MODIFIED SELBERG INTEGRAL of f

$$\widetilde{J}_f(N,H) \stackrel{def}{=} \sum_{N<x\le 2N} \Big| \frac{1}{H} \sum_{h\le H} \sum_{x-h<n<x+h} f(n) - M_f(x,H) \Big|^2$$

where $M_f(x,H)$ is the same mean-value above in $J_f(N,H)$.

Defining the CESARO WEIGHT,

$$C(a) \stackrel{def}{=} \max\left(1 - \frac{|a|}{H}, 0\right),$$

we see that the modified Selberg integral of f is

$$\widetilde{J}_f(N,H) = \sum_{N<x\le 2N} \Big| \sum_n C(n-x) f(n) - M_f(x,H) \Big|^2.$$

In fact, these three integrals are particular instances of so-called "WEIGHTED SELBERG INTEGRAL" of f,

$$J_{w,f}(N,H) \stackrel{def}{=} \sum_{N<x\le 2N} \Big| \sum_n w(n-x) f(n) - M_f(x,w) \Big|^2,$$

that we (with Laporta) [CL] introduced in Generations. (We'll refer to "Generations", here, `arxiv`'s 1205.1706v3.)

Here $w : [-H,H] \to \mathbb{C}$ is absolutely bounded and $M_f(x,w)$ depends on w. Selberg integral and symmetry integral have w equal to "u", characteristic function of $(0,H]$ and to sign function "sgn", resp.; modified Selberg integral comes with the Cesaro weight C.

For these weights $w : [-H, H] \to \mathbb{C}$ the MASS is

$$\widehat{w}(0) \stackrel{def}{=} \sum_a w(a)$$

and we define, for $f(n) = \sum_{d|n} g(d)$, the ARITHMETIC FORM (see Generations' §2) of s.i. mean-value as

$$M_f(x, w) \stackrel{def}{=} \widehat{w}(0) \sum_q \frac{g(q)}{q},$$

compare Generations' §3. We call $g(q) \stackrel{def}{=} \sum_{d|q} \mu(d) f(q/d)$ the ERATOSTHENES TRANSFORM of f.

Actually, in case $f = d_k$ things are more complicated and s.i. mean-value is defined in terms of residues, say, by ANALYTIC FORM; also the arithmetic form depends strongly on x so that our Generations' "$k-$folding method", say, proves proximity of these two forms.

However, now on, for clarity we will confine to functions $f = g * \mathbf{1}$ (above), where g nor its support depend on x. I started study of these, say, "SIEVE FUNCTIONS", in the 2000s. For them, the s.i. mean-value doesn't depend on x so may be defined as above.

Methods used for these sieve functions in the case of Selberg integral and symmetry integral successfully adapt to a class of more general weights. We'll see this in the Theorems 2 & 3, following.

I start with a very recent result : it roughly says that a *non-trivial symmetry* integral *is "inherited", from Eratosthenes transform* g, to $f = g * \mathbf{1}$. Proof uses Large Sieve.

We will abbreviate $H \asymp N^\Theta$ for $N^\Theta \ll H \ll N^\Theta$, say, Θ is the WIDTH of the short interval $[x-H, x+H]$. Also, write $h \asymp N^\theta$ for $N^\theta \ll h \ll N^\theta$, say, θ is the width of shorter interval $[x-h, x+h]$.

We abbreviate $g \lll 1$ and say g is "essentially bounded", when $\forall \varepsilon > 0 \ g(n) \ll_\varepsilon n^\varepsilon$, $n \to \infty$.

We link the symmetry integral of our $f = g * \mathbf{1}$, to the symmetry integral of its Eratosthenes transform, g.

Theorem 1. *Fix* $\Theta \in (1/3, 1)$, *let* $H \asymp N^\Theta$ *as* $N \to \infty$. *Let* $g : \mathbb{N} \to \mathbb{R}$ *be essentially bounded. Then* $\exists G \in (0,1)$:

$$\sum_{x \sim N} \Big| \sum_{x-h \le q \le x+h} \mathrm{sgn}(q-x) g(q) \Big|^2 \ll N h^{2-G}, \ \ \forall \theta \in (0, \Theta]$$

$$\Downarrow$$

$$\sum_{x \sim N} \Big| \sum_{x-H \le n \le x+H} \mathrm{sgn}(n-x) (g * \mathbf{1})(n) \Big|^2 \ll N H^{2-G'},$$

where $G' > 0$ *depends on* $G > 0$ *and* Θ.

Thus, we have inheritance of non-trivial symmetry integral, from g to $g * \mathbf{1}$ (from f Eratosthenes transform to f).

Unfortunately, the width here has to be $\Theta > 1/3$, due to technique (Large Sieve inequality) used in the Proof. This is rather elementary, since it applies only Cauchy inequality.

Our second result is even more elementary (not even a Cauchy inequality!), while our third applies state-of-the-art very sophisticated techniques (due to Duke, Friedlander & Iwaniec, [DFI], for bilinear forms of Kloosterman fractions).

We define the CORRELATION $\mathcal{C}_w$ of $w : [-H, H] \to \mathbb{C}$ as

$$\mathcal{C}_w(a) \stackrel{def}{=} \sum_{h_2 - h_1 = a}\sum w(h_2)\overline{w(h_1)}$$

and, then, we call w an ARITHMETIC weight when its correlation satisfies

$$(A) \qquad \sum_{a \equiv 0 (\bmod\ \ell)} \mathcal{C}_w(a) = \frac{1}{\ell}\sum_{a} \mathcal{C}_w(a) + O(H), \quad \forall \ell \le 2H$$

For example, our weights u, sgn, C are arithmetic.

Furthermore, we need another definition:

$$w \text{ is GOOD } \stackrel{def}{\Longleftrightarrow} w \text{ is arithmetic \& absolutely bounded}$$

i.e., "*absolutely bounded*", here, means $|w| \le K$ and $K > 0$ is absolute (say, $1, 2, 4$), independent of H.

For example, our main weights u, sgn, C are good.

The "modified Vinogradov notation" (used by G. Kolesnik)

$$A \lll B \stackrel{def}{\Longleftrightarrow} A \ll_\varepsilon N^\varepsilon B, \quad \forall \varepsilon > 0$$

is very useful to save arbitrary small powers.

We have a general elementary result (doesn't even use Cauchy inequality), for weighted Selberg integrals, with good and real w, of essentially bounded arithmetic functions f.

Theorem 2. *Let* $N, H \in \mathbb{N}$, $H \to \infty$, $H = o(N)$ *as* $N \to \infty$. *Assume* $Q = Q(N, H) \to \infty$ *and* $g : \mathbb{N} \to \mathbb{R}$ *is an arithmetic function with support inside* $[1, Q]$, *abbreviate* $f = g * \mathbf{1} \lll 1$. *Then, for all good weights* $w : [-H, H] \to \mathbb{R}$,

$$J_{w,f}(N, H) \lll NH + Q^2 H + QH^2 + H^3.$$

We give a very brief sketch of the Proof.

- We apply a kind of elementary Dispersion Method (say, we open the square and exchange sums): then our weighted Selberg integral is a correlations' average.
- Main terms are cancelled out (+negligible remainders). (Here we use hypothesis: w is good.)
- For other terms, a positivity argument for the "discrete Fourier transforms" of correlations (whose DFT are in fact always non-negative: immediate).

We we will not sketch the Proof of next result, but only say that it runs approximately as this; except, that last step is substituted by an application (via fractional parts' finite Fourier expansions) of [DFI] quoted results for averages of Kloosterman sums (in turn, they, after many other tools, apply Weil's bound for Kloosterman sums stemming from "Weil's RH for curves over finite fields").

We state a companion result, applying a very technical result on bilinear forms of Kloosterman fractions [DFI].

Theorem 3. *Fix* $0 < \theta < 1/2$. *Let* $N, H \in \mathbb{N}$, $H \asymp N^{\theta}$, *as* $N \to \infty$. *Assume* $Q = Q(N,H) \to \infty$ *and* $g : \mathbb{N} \to \mathbb{R}$ *is an arithmetic function with support inside* $[1, Q]$, *abbreviate* $f = g * \mathbf{1} \lll 1$. *Then, for all good weights* $w : [-H, H] \to \mathbb{R}$ *and* $\forall \delta > 0$,

$$J_{w,f}(N,H) \lll NH + N^{\delta} Q^{95/48} H^2 + N^{1-2\delta/3} H^2 + QH^2.$$

Thus we may summarize Theorem 2 and Theorem 3 in the following.

Corollary. *Fix* $0 < \theta < 1/2$. *Let* $N, H \in \mathbb{N}$, $H \asymp N^{\theta}$, *as* $N \to \infty$. *Assume* $Q \asymp N^{\lambda}$, *with, say,* LEVEL $\lambda = \lambda(\theta)$ *in the range* $\lambda \in (0, \max(1/2 + 1/190, (1+\theta)/2))$ *and* $g : \mathbb{N} \to \mathbb{R}$ *is an arithmetic function with support inside* $[1, Q]$, *abbreviate* $f = g * \mathbf{1} \lll 1$. *Then, for all good weights* $w : [-H, H] \to \mathbb{R}$, $\exists \delta = \delta(\theta, \lambda) > 0$:

$$J_{w,f}(N,H) \ll NH^2 / N^{\delta}.$$

See that usual approach to correlations via the arithmetic progressions suffers from the "level 1/2 barrier", due to the Large Sieve inequality. Our new approach through finite Fourier expansions and [DFI] results enables us to bypass this barrier: we reach level 1/2 + 1/190 in Corollary (of course, not for arithmetic progressions !) .

Remark. An application is to the well-known "truncated von Mangoldt functions", with parameter R, introduced by Goldston - Pintz - Yildirim (say $Q = R \asymp N^{1/2+1/190-\varepsilon}$). In fact, our $f = g * \mathbf{1}$ plays the rôle of their Λ_R.

As regards Theorem 1, not only symmetry is inherited, but also proximity of Selberg integral & the modified one (we consider weight $C - u$, has mass 0, then Cauchy inequality).

As we saw, there are a variety of techniques to bound weighted Selberg integrals, many to come, too...
Here is a "short bibliography", for the talk. (By the way, see the ArXiV for the present results to come !)

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THANK YOU !!!