

# On the Selberg integral of the $k$ -divisor function and the $2k$ -th moment of the Riemann zeta function

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## 1. Introduction.

We will link the two problems, estimating the  $2k$ -th moment of the RIEMANN  $\zeta$ -FUNCTION (on the CRITICAL LINE  $\sigma = \frac{1}{2}$ )

$$I_k(T) \stackrel{def}{=} \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt$$

and finding non-trivial bounds for the, say, SELBERG INTEGRAL of the  $k$ -divisor function,  $d_k(n)$  (that has Dirichlet series  $\zeta^k$ )

$$J_k(x, h) \stackrel{def}{=} \int_{hx^\varepsilon}^x \left| \sum_{t < n \leq t+h} d_k(n) - M_k(t, h) \right|^2 dt$$

where, say,  $M_k(t, h)$  is the “expected value” of the (inner) sum. This gives  $d_k$  over the (say) “SHORT INTERVAL”  $[t, t + h]$  (as  $h = o(t) \forall t \in [hx^\varepsilon, x]$ ); here and in the sequel  $\varepsilon > 0$  will be arbitrarily small, not the same at each occurrence.

Actually, Ivić already gave (in a paper to appear on JTNB for JAXXV Proc.) a non-trivial bound for  $J_k(x, h)$  when the WIDTH of the s.i. (abbrev. short interval), NAMELY  $\theta := \frac{\log h}{\log x}$  is greater than  $\theta_k \stackrel{def}{=} 2\sigma_k - 1$  (with  $\sigma_k$  CARLSON’S ABSCISSA):

$$(Ivić, JTNB) \quad \theta > \theta_k \Rightarrow \exists \delta = \delta(k) > 0 : J_k(x, h) \ll \frac{xh^2}{x^\delta}$$

(see the trivial bound :  $J_k(x, h) \ll xh^2(\log x)^c$ ,  $c = c(k)$ , here)  
 For example,  $\theta_3 = \frac{1}{6}$ ,  $\theta_4 = \frac{1}{4}$ ,  $\theta_5 = \frac{11}{30}$ , ... (from values of  $\sigma_k$ ).

This result clearly gives non-trivial bounds for  $J_k$ , using  $\zeta$ -moments information (off the critical line, into the  $\sigma_k$ ).

So, knowledge of the  $\zeta$  implies knowledge of the  $d_k$  in A.A. (abbrev. almost all) the S.I. (short intervals).

$$(\text{See: } J_k \text{ non trivial} \Rightarrow \sum_{t < n \leq t+h} d_k(n) \sim M_k(t, h), \text{ A.A.S.I.})$$

However, we can also go in the opposite direction: if we have some kind of non-trivial information about the  $d_k$ , we can improve our knowledge (at least, on the  $2k$ -th moments) of the Riemann  $\zeta$ -function. Actually, this idea is due to Ivić, who gave a link between the “(AUTO-)CORRELATION” of  $d_k$  with “SHIFT-PARAMETER”  $a$ , namely

$$\mathcal{C}_k(a) \stackrel{\text{def}}{=} \sum_{n \leq x} d_k(n)d_k(n+a) \quad a \in \mathbb{N} \text{ (here } x \in \mathbb{N}, x \rightarrow \infty)$$

(the SHIFT is a positive integer, here; however,  $\mathcal{C}_k(-a)$  is close enough to  $\mathcal{C}_k(a)$ ; also,  $\mathcal{C}_k(0)$  is relatively easy to compute).

The problem of estimating  $d_k$  (AUTO-)CORRELATION,  $\mathcal{C}_k(a)$ , at least for a FIXED  $a \neq 0$  ( $a = 0$  is “TRIVIAL”) is called the  $k$ -ARY ADDITIVE DIVISOR PROBLEM ( $k$ -ARY ADD. DIV. PBM); the trivial case  $k = 1$  (with  $\mathcal{C}_k(a) = x \forall a \in \mathbb{Z}$ ) and the “BINARY ADDITIVE DIVISOR PROBLEM”,  $k = 2$ , are the only two solved pbms, here.

Case  $k = 3$  is the TERNARY ADDITIVE DIVISOR PROBLEM (sometimes called “LINNIK PROBLEM”, but Yu.V.Linnik has left many unsolved pbms !): some time ago, Vinogradov & Takhtadjan (right spelling ?) announced its solution but with, as yet, unfilled holes in their (extremely technical !) “proof”.

This approach, actually, still suffers from our lack of information about  $SL(\mathbb{Z}, 3)$ ; while our (enough good) state of the art about, instead,  $SL(\mathbb{Z}, 2)$  allowed Ivić, Motohashi and Jutila to solve satisfactorily the binary additive divisor problem (but, different approaches work; however, with weaker remainders).

Thus, so far, no one has proved (for  $k > 2$ ), given  $a \in \mathbb{N}$ ,

$$(*)_k \quad \mathcal{C}_k(a) = xP_{2k-2}(\log x) + \Delta_k(x, a), \quad \Delta_k(x, a) = o(x),$$

AS  $x \rightarrow \infty$  (THE  $k$ -ARY ADDITIVE DIVISOR PROBLEM), not even for a single shift  $a > 0$  (already  $k = 2$  has delicate “ $a$ -UNIFORMITY” issues: following).

(MAIN TERM HAS  $P_{2k-2}(\log x)$ , A  $2k - 2$  DEG. POLY. IN  $\log x$ )

Here it comes into play the idea of Ivić (see his paper in Palanga 1996 Conference Proc.) of LINKING the estimate of the  $2k$ -TH MOMENT,  $I_k(T)$ , TO A SUM OF CORRELATIONS  $\mathcal{C}_k(a)$  performed OVER  $a$  (THE SHIFT), up to (roughly, we avoid technicalities), say,  $h := \frac{x}{T}$  (the S.I. comes in !)

In order to be more precise, we NEED also TO ABBREVIATE:

$$A \lll B \stackrel{def}{\iff} \forall \varepsilon > 0 \quad A \ll_{\varepsilon} x^{\varepsilon} B$$

(with  $x, X$  or even  $T$  our “MAIN VARIABLES”, indep. &  $\rightarrow \infty$ ) I.E., the MODIFIED VINOGRADOV NOTATION  $\lll$  allows us to IGNORE ALL THE ARBITRARILY SMALL POWERS; also, we’ll say that the ARITHMETIC FUNCTION (A.F.)  $f : \mathbb{N} \rightarrow \mathbb{R}$  is ESSENTIALLY BOUNDED, write  $f \lll 1$ , when  $\forall \varepsilon > 0 \quad f(n) \ll_{\varepsilon} n^{\varepsilon}$  (as  $n \rightarrow \infty$ ). For example, all the  $d_k$  ( $\forall k \in \mathbb{N}$ ) are ess.bd :

$$\forall k \in \mathbb{N} \quad d_k \lll 1$$

whence they contribute individually small powers (ignored); Shiu estimates (see  $J_k$  triv.est.quoted above), a kind of Brun-Titchmarsh for (suitable multiplicative A.F., like)  $d_k$ , let these give, on average over (all) s.i., powers of  $\log$ . By the way,

$$L := \log x \quad (\text{or } L := \log X, \text{ even } L := \log N)$$

is the abbreviation for the logarithm of our main variable.

For example, the (conjectured,  $\forall k > 2$ ) M.T. of  $(*)_k$  is

$$xP_{2k-2}(\log x) \ll_k xL^{2k-2} \lll x.$$

Here, it seems that the first to propose explicitly this form for  $(*)_k$  is Ivić, who also gave explicitly the polynomial  $P_{2k-2}$ , that is essentially bounded (w.r.t.  $x$ ). However, as we'll see in a moment, it depends, also, on the SHIFT  $a > 0$ .

We'll give now, avoiding technicalities, Ivić' s argument.

After some work (expand the square & mollify, take relevant ranges, ...) he gets that  $I_k(T)$  is

$$I_k(T) = I_k''(T) + \mathcal{O}_\varepsilon(T^\varepsilon T)$$

with

$$I_k'' \stackrel{\text{def}}{=} \frac{1}{M} \sum_{a \leq M^{1+\varepsilon} T^{-1}} \sum_{M < n \leq M'} d_k(n) d_k(n+a) \int_{\frac{T}{2}}^{2T} \phi(t) e^{ita/n} dt,$$

where  $M < M' \leq 2M$ , with  $M \lll T^{k/2}$ , say  $h \lll M/T$ , the smooth (i.e.,  $C^\infty$ ) test-function  $\phi$  has support into  $]T/2, 2T[$ ,  $\phi([3T/4, 4T/3]) \equiv 1$ , and has good decay

$$\phi^{(R)}(t) \ll_R T^{-R}, \quad \forall R \in \mathbb{N}.$$

We should keep in mind, here, that for fixed  $k \in \mathbb{N}$  we seek

$$I_k(T) = \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt \lll T \quad (\text{“}2k\text{--TH MOMENT PBM”})$$

that (for  $k > 2$ ) is our aim; in fact, English school gave first 2 cases: Ingham in 1926 asymptotics for  $k = 1$  (not too difficult!) and  $k = 2$  (for both, he gave only  $P_{2k-2}$  leading term, hence error  $\log x$  better than main term); then, Heath-Brown in 1979 gave, for  $k = 2$  (solved 2–add.div.pbm. using Weil’s bound for Kloosterman sums),  $P_6$  (not explicitly!) plus error  $E_2 \lll T^{7/8}$ .

Starting from ’94 & ’95, a series of Ivić & Motohashi papers (applying  $SL(\mathbb{Z}, 2)$  considerations for the binary add.div.pbm.) gave  $E_2 \lll T^{2/3}$  and, in mean-square, even  $E_2 \lll \sqrt{T}$ . (Here log-pows, not small pows!). Ivić explicited  $P_{2k-2}$  (when  $k = 2$ ). Like the binary add.div.pbm., this is not the whole story !

The case  $k = 3$ , again (recall  $\mathcal{C}_3(a)$  PBM), is unsolved. The bound

$$I_3(T) \lll T$$

is called THE “SIXTH MOMENT” PBM (actually, this is the weak version) & has a LINK (in Ivić, Proc. Cardiff 1996 Symposium) WITH the TERNARY ADDITIVE DIVISOR PBM.

Another interesting moment (Heath-Brown ’79):  $I_6 \ll T^2 L^c$ .

One glimpse,  $2k$ –th pbm. ( $k \leq 2$ , see Ivić & Motohashi, quoted): PREDICTED ASYMPTOTICS  $I_k \sim C(k)TL^{k^2}$ ,  $\forall k \geq 1$  (English school, again!) applying RANDOM MATRIX THEORY (Quantum Physics concepts inspiration!) in (2000s) seminal works of J. Keating & N. Snaith (at Bristol); many others (Conrey, Ghosh to name two). The RMT– $\zeta$  link originated (Dyson-Montgomery coffee-break) in 1972. (Sorry for brevity.)

Now on (that's why we mentioned weak  $2k$ -th pbm !) we can ignore (in bounds for  $I_k$ ) all terms which are  $\lll T$ .

We give an idea of the polynomial,  $P_{2k-2}$ , given by Ivić, before to proceed. It's (see Palanga Conference Proc. for details)

$$P_{2k-2}(\log x) \stackrel{def}{=} \frac{1}{x} \int_0^x \sum_{q=1}^{\infty} \frac{c_q(a)}{q^2} R_k^2(\log t) dt,$$

with, say,  $R_k(\log t) \stackrel{def}{=}$

$$\frac{C_{-k}(q)}{(k-1)!} \log^{k-1} t + \frac{C_{1-k}(q)}{(k-2)!} \log^{k-2} t + \dots + \frac{C_{-2}(q)}{1!} \log t + C_{-1}(q)$$

depending on  $q$ , but not on  $a$  (this is vital); also, w.r.t.  $x$ ,  $R_k(\log t) \lll 1$  and this is very useful ! We'll see in a moment that the shape of these  $C_j(q)$  is important only in case  $q = 1$ . By the way, here  $c_q(a)$  is the RAMANUJAN SUM, defined as ( $\sum^*$  is over  $q$ -coprime  $js$ )

$$c_q(a) \stackrel{def}{=} \sum_{j \pmod{q}}^* e_q(ja) = \sum_{\substack{d|q \\ d|a}} d\mu\left(\frac{q}{d}\right)$$

Hence, say,  $S(a) \stackrel{def}{=} \max(0, h - |a|)$  ( $\hat{S}$  is Fejér's kernel) gives

$$\hat{S}\left(\frac{j}{q}\right) \stackrel{def}{=} \sum_a S(a) e_q(ja) \geq 0 \Rightarrow \sum_a S(a) c_q(a) \geq 0$$

(see the link with  $J_k$  soon) and from the elementary,  $\forall d \in \mathbb{N}$ ,

$$\sum_{\substack{a \\ a \equiv 0 \pmod{d}}} S(a) = h + 2 \sum_{b \leq h/d} (h - db) = \frac{h^2}{d} + d \left\{ \frac{h}{d} \right\} \left( 1 - \left\{ \frac{h}{d} \right\} \right),$$

we get (apply  $c_q(a)$ , above), writing  $\mathbf{1}_\wp = 1$  if  $\wp$  holds,  $= 0$  else:

$$\sum_a S(a)c_q(a) = \mathbf{1}_{q=1}h^2 + \sum_{d|q} d^2\mu\left(\frac{q}{d}\right) \left\{\frac{h}{d}\right\} \left(1 - \left\{\frac{h}{d}\right\}\right).$$

(It is here evident  $q = 1$  greater importance.) Thus, (see Ivić):

$$(1) \quad \sum_a S(a)xP_{2k-2}(\log x) = h^2 \int_{hx^\varepsilon}^x R_k^2(1, \log t)dt + \text{TAILS},$$

where we mean, by “TAILS”, remainders which are  $\lll h^3$ . Here, the part of  $R_k(\log t)$  term with  $q = 1$  is, say,  $R_k(1, \log t) \stackrel{def}{=}$

$$\frac{C_{-k}(1)}{(k-1)!} \log^{k-1} t + \frac{C_{1-k}(1)}{(k-2)!} \log^{k-2} t + \dots + \frac{C_{-2}(1)}{1!} \log t + C_{-1}(1)$$

and equals the term  $M_k(\log t)$  into the Selberg integral; as it should be, since (from an elementary version of the Linnik’s Dispersion method), assuming  $(*)_k$  with this  $P_{2k-2}$ , we get

$$(2) \quad J_k(x, h) \sim \sum_a S(a)\mathcal{C}_k(a) - h^2 \int_{hx^\varepsilon}^x M_k^2(\log t) dt \\ \sim \sum_a S(a)\Delta_k(x, a),$$

where  $\sim$  means ignoring “TAILS” (see above) & “DIAGONALS”, i.e. remainders  $\lll xh$ . We remark that BOTH THESE KIND OF ERRORS ARE NEGLIGIBLE, since they both contribute  $\lll T$  to the final result for  $I_k(T)$ . (This can be easily seen.)

Then, due to  $I_k''$  expression, Ivić made a hypothesis about (avoiding technicalities) sums of  $\Delta_k(x, a)$  (remainders into  $(*)_k$  above), performed over the shift  $a$ , say  $G_k$ , which implies the bound  $I_k(T) \lll T$  (for the same  $k > 2$ ). NOW ON  $k > 2$ .

Of course, he doesn't need  $(*)_k$  to hold INDIVIDUALLY  $\forall a$  ( $\leq h$ , here), but he observes that he's summing up, into  $G_k$ , WITHOUT the modulus over the remainder,  $\Delta_k(x, a)$ , so some  $a$ -cancellation can take place, here.

So far, he passes from an asymptotic formula  $(*)_k$  to an  $a$ -averaged form of it, which is easier to prove (however, yet nobody has done it !).

Here, with applications in mind,

WE PASS FROM A SINGLE AVERAGE TO A DOUBLE AVERAGE

Building on his expression for  $I_k''$ , it's possible to make a less stringent hypothesis, to HAVE A MORE FLEXIBLE PROCEDURE for the remainders  $\Delta_k(x, a)$ .

We use, also, our previous works on the Selberg integral of the A.F.  $f$  (ESSENTIALLY BOUNDED & REAL) in order to let the Selberg integral of  $d_k$ , i.e.  $J_k(x, h)$ , come into play. (It is a kind of "DOUBLE AVERAGE" of  $\Delta_k(x, a)$ .)

Unfortunately, due to an exponential factor multiplying  $d_k(n)d_k(n+a)$  into  $\mathcal{C}_k(a)$  we can't get a link with  $I_k(T)$  using only  $J_k(x, h)$  (WITH  $h \lll \frac{x}{T}$ ,  $x \lll T^{k/2}$ ), but we need, also, to make an hypothesis on another double average of remainders  $\Delta_k(x, a)$ . We will give our Theorem and a sketch of the proof.



## 2. Result and Remarks.

THEOREM. Let  $M < M' \leq 2M$ ,  $T^{1+\varepsilon} \leq M \ll T^{k/2}$  and  $H = M^{1+\varepsilon}/T$ , with double average  $\tilde{G}_k = \tilde{G}_k(M, T)$  defined as

$$\tilde{G}_k \stackrel{def}{=} \sup_{M \leq x \leq M', t \leq H} \left( \frac{1}{t} J_k(x, t) + \frac{1}{t} \left| \sum_{h \leq t} \sum_{h < a \leq t} \Delta_k(x, a) \right| \right).$$

Then for  $k = 3, 4$  and any fixed  $\varepsilon > 0$  we have

$$I_k(T) \ll_{k, \varepsilon} T^{1+\varepsilon} \left( 1 + \sup_{T \ll M \ll T^{k/2}} \tilde{G}_k(M, T)/M \right).$$

We remark that, in spite of the fact that Ivić's Theorem holds  $\forall k > 2$  (integer), we have some trouble in handling Selberg's integral TAILS, since they contribute to  $\tilde{G}_k$  as (in the sup above)

$$\lll \frac{1}{t} t^3 \lll H^2 \lll \frac{M^2}{T^2}$$

which gives to  $I_k(T)$  a contribute (other sup above)

$$\lll T \left( \frac{M^2}{T^2} M^{-1} \right) \lll \frac{M}{T} \lll T^{k/2-1}$$

that is  $\lll T$  ONLY WHEN  $k/2 \leq 2$ , I.E.  $k \leq 4$  HERE.

We trust the possibility to have a link as above not only for the sixth & the eighth moment, but the TAILS arise naturally when applying the Linnik method and even a more careful analysis will almost surely not eliminate them! While they're negligible for Selberg integral, not so for present application!

In order to give an idea of the proof, we'll avoid technics and highlight the ideas (few & elementary) involved.

First of all, since Ivić already has a sum of remainders, SAY

$$\sum_{a \leq t} \Delta_k(x, a)$$

we simply let the ARITHMETIC MEAN appear (together with the other double sum, see  $\tilde{G}_k$  definition). Then, for this, see that

$$\frac{1}{t} \sum_{h \leq t} \sum_{a \leq h} \Delta_k(x, a)$$

(a kind of average of the above, something like  $C^1$  process in Fourier series) can be expressed as (exchanging sums)

$$\frac{1}{t} \sum_{a \leq t} (t - a + 1) \Delta_k(x, a) = \frac{1}{t} \sum_{a \leq t} (t - a) \Delta_k(x, a) + \frac{1}{t} \sum_{a \leq t} \Delta_k(x, a)$$

which reduces to (using  $\Delta_k(x, a) \lll x$ )

$$\sim \frac{1}{t} \sum_{a \leq t} (t - a) \Delta_k(x, a) \sim \frac{1}{2t} \sum_{0 \leq |a| \leq t} (t - |a|) \Delta_k(x, a),$$

(+ diag.s & tails); and  $S(a) = \max(t - |a|, 0)$ ,  $\forall 0 \leq |a| \leq t \Rightarrow$

$$\sum_{0 \leq |a| \leq t} (t - |a|) \Delta_k(x, a) \sim J_k(x, t)$$

we get the desired bound with the Selberg integral (& double average).

We remark, in passing, that “additional” double average can't be dispensed with.