ON SOME LOWER BOUNDS OF SOME SYMMETRY INTEGRALS

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Abstract. We study the "symmetry integral", say I_f, of some arithmetic functions $f:\mathbb{N}\to\mathbb{R}$; we obtain from lower bounds of I_f (for a large class of arithmetic functions f) lower bounds for the "Selberg integral" of f, say J_f (both these integrals give informations about f in almost all the short intervals $[x-h,x+h]$, when $N\le x\le 2N$). In particular, when $f=d_k$, the divisor function (having Dirichlet series ζ^k, with ζ the Riemann zeta function), where $k\ge 3$ is integer, we give lower bounds for the Selberg integrals, say $J_k=J_{d_k}$, of the d_k. We apply elementary methods (Cauchy inequality to get Large Sieve type bounds) in order to give I_f lower bounds.

1. Introduction and statement of the results.

We give lower bounds of SYMMETRY INTEGRALS (here $\mathrm{sgn}(0)\buildrel{def}\over{=}0$, $r\neq 0\Rightarrow \mathrm{sgn}(r)\buildrel{def}\over{=}\frac{r}{|r|}$)

$$I_f(N,h)\buildrel{def}\over{=}\int_N^{2N}\Big|\sum_{|n-x|\le h}\mathrm{sgn}(n-x)f(n)\Big|^2dx$$

for a large class of arithmetic functions $f:\mathbb{N}\to\mathbb{R}$. For a motivation to study I_f, see esp. [C].

A related integral is the, say, SELBERG INTEGRAL, defined as

$$J_f(N,h)\buildrel{def}\over{=}\int_N^{2N}\Big|\sum_{x<n\le x+h}f(n)-M_f(x,h)\Big|^2dx,$$

where the MEAN-VALUE $M_f(x,h)$ depends "weakly" on x and is expected to depend linearly on h (esp., it's of the kind h times a polynomial in $\log x$, see the following). It is a kind of "MAIN TERM" of the sum IN THE "SHORT INTERVAL" $[x,x+h]$ (i.e., $h=o(x)$); so, we may expect it to approximate ($x\in\mathbb{N}$, $x\to\infty$ here), when $f=g*\mathbf{1}$, (compare [C1])

$$h\Big(\frac{1}{x}\sum_{n\le x}f(n)\Big)=\frac{h}{x}\sum_d g(d)\left[\frac{x}{d}\right]\approx h\sum_{d\le x}\frac{g(d)}{d}.$$

While the former integral measures the ALMOST-ALL (i.e., for all $N\le x\le 2N$, except $o(N)$ of them) symmetry (around x) of f in the SHORT (since $h=o(x)$) INTERVAL $[x-h,x+h]$, the Selberg integral gives an "AVERAGE VALUE" to f in $[x,x+h]$, for A.A. (abbrev. almost all, s.i. shortens short intervals) these S.I.

Actually, it is a matter of evidence that knowing the (average) values of f into A.A.S.I. gives immediate information about the relative symmetry of f; however, let's go into more precise details and let's give an EXPLICIT CONNECTION between these two integrals:

$$I_f(N,h)=\int_N^{2N}\Big|\sum_{x<n\le x+h}f(n)-\sum_{x-h\le n<x}f(n)\Big|^2dx\ll\int_N^{2N}\Big|M_f(x,h)-M_f(x-h,h)\Big|^2dx+\int_N^{2N}|f(x)|^2dx+$$

$$+\int_{N-h}^{2N-h}|f(x)|^2dx+\int_N^{2N}\Big|\sum_{x<n\le x+h}f(n)-M_f(x,h)\Big|^2dx+\int_N^{2N}\Big|\sum_{x-h<n\le x}f(n)-M_f(x-h,h)\Big|^2dx;$$

and, using the "MODIFIED VINOGRADOV NOTATION", i.e. (in general, $F:\mathbb{N}\to\mathbb{C}$, here)

$$F(N,h)\lll G(N,h)\ \buildrel{def}\over{\Longleftrightarrow}\ \forall\varepsilon>0\quad |F(N,h)|\ll_{\varepsilon}N^{\varepsilon}G(N,h),$$

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so to leave (arbitrarily) small powers, ASSUMING NOW ON f ESSENTIALLY BOUNDED, i.e. $f(n) \lll 1$, we have

$$I_f(N,h) \ll_{\varepsilon} J_f(N,h) + \int_{N-h}^{2N-h} \Big| \sum_{x<n\le x+h} f(n) - M_f(x,h)\Big|^2 dx + \int_N^{2N} \Big|M_f(x,h) - M_f(x-h,h)\Big|^2 dx + N^{1+\varepsilon}$$

$$\ll_{\varepsilon} J_f(N,h) + \int_N^{2N} \Big|M_f(x,h) - M_f(x-h,h)\Big|^2 dx + N^{\varepsilon}(N+h^3),$$

where the last remainder comes from "TAILS", i.e. terms $\lll h^3$ (see that $M_f(x,h) \lll h$ is a consequence of the previous remarks on M_f and $f \lll 1$). We may assume, of course, that the difference $M_f(x,h) - M_f(x-h,h)$ is A.A. small (i.e., its mean-square is "small"), due to the fact (compare the above, about M_f choice) that M_f is "weakly" dependent on x (like the case M_k following, for $f = d_k$ with generating Dirichlet series ζ^k).

Then, ignoring these contributes together with the negligible $\lll N + h^3$, we derive a lower bound of J_f, starting from a lower bound of I_f, here.

(We'll give a more precise calculation, following, for the more interesting cases $f = d_k$, see the above.)

We start simply remarking that the definition of "MIXED SYMMETRY INTEGRALS" (compare [C5]):

$$I_{f,f_1}(N,h) \stackrel{def}{=} \int_N^{2N} \sum_{|n-x|\le h} \mathrm{sgn}(n-x) f(n) \sum_{|m-x|\le h} \mathrm{sgn}(m-x) f_1(m) dx$$

allows us to give a lower bound to I_f, applying (expand the inner square), abbrev. I_f for $I_f(N,h)$ & similia,

$$0 \le I_{f-f_1} = I_f - 2I_{f,f_1} + I_{f_1}$$

to get ($\forall N, h$ which are feasible)

$$I_f(N,h) \ge 2I_{f,f_1}(N,h) - I_{f_1}(N,h). \tag{1}$$

Here (1) is true $\forall f, f_1 : \mathbb{N} \to \mathbb{R}$ (ANY COUPLE OF REAL ARITHMETIC FUNCTIONS).

However, in order to give a non-trivial lower bound to I_f, we need I_{f_1} to be "*smaller*" than $2I_{f,f_1}$. (That's the reason why we will give our general lower bound for "mixed" integrals, but not for "pure" ones.)

It will turn out, from our general result (next Theorem, compare the Lemma at next section), that the choice $f = d_k$ (general $k-$divisor function) and $f_1 = d$ (i.e., $k = 2$, divisor function) gives non-trivial lower bounds for d_k symmetry integral; then, previous connection implies lower bounds for its Selberg integral, $J_k := J_{d_k}$. This lower bound for J_k is (ignoring logarithms) of the same order of magnitude of the diagonal (compare [C4], where this order of magnitude is required as an upper bound, to treat $2k-$th moments of ζ).

In order to simplify the exposition, we need to compare our variables to our MAIN VARIABLE, i.e. $N \to \infty$, from the point of view of exponents, using, say, $L := \log N$ ("logarithmic scale") :

a) $\theta := \frac{\log h}{L}$ is the WIDTH (not the length, that's h) of the short interval $[x, x+h]$ (say, also of $[x-h, x+h]$);
b) $\lambda := \frac{\log Q}{L}$ is the LEVEL (see §1 in [C3]) of $f : \mathbb{N} \to \mathbb{R}$, $f = g * \mathbf{1}$, $g(q) = 0$, $\forall q > Q$;
c) $\delta := \frac{\log D}{L}$ is the "auxiliary level" of our mixed integral (in the following Theorem).

We explicitly remark that ANY INEQUALITY involving these quantities WILL BE implicitly assumed to be SHARP (compare §1 of [C3]): esp., our width will always be positive (i.e., $\exists \varepsilon_0 > 0$, absolute, with $\theta > \varepsilon_0 > 0$.).

Our methods are elementary, as we apply a kind of LARGE SIEVE INEQUALITY, using the SPACING PROPERTY OF FAREY FRACTIONS (see the Lemma at next section).

We indicate, as usual, the DISTANCE TO INTEGERS (of any $\alpha \in \mathbb{R}$) as $\|\alpha\| := \min_{n\in\mathbb{Z}} |\alpha - n|$.

Our results are the following.

Theorem. *Fix* width $0<\theta<1/2$, level $0<\lambda<1$ *and* "auxiliary level" δ, *with* $\theta<\delta<\lambda$ *and* $\delta+\lambda<1$. *Let* $N,h,D,Q\in\mathbb{N}$, *with* $h=[N^\theta]$, $D=[N^\delta]$, $Q=[N^\lambda]$. Assume $g_1:\mathbb{N}\to\mathbb{R}$, $g:\mathbb{N}\to\mathbb{R}$ *supported (resp.) in* $[1,D]$, $[1,Q]$, *with both* $1\le g_1\lll 1$ *and* $1\le g\lll 1$; *set* $f_1:=g_1*\mathbf{1}$, $f:=g*\mathbf{1}$. *Then, defining the* Ramanujan coefficients *of an* essentially bounded *arithmetic function* $F:\mathbb{N}\to\mathbb{C}$ *as*

$$R_\ell(F)\overset{def}{=}\sum_{\substack{m=1\\ m\equiv 0(\ell)}}^{\infty}\frac{(F*\mu)(m)}{m}=\frac{1}{\ell}\sum_{n=1}^{\infty}\frac{G(\ell n)}{n}\lll\frac{1}{\ell},$$

where, say, $G\overset{def}{=}F*\mu\lll 1$ has finite support (so *to ensure* absolute convergence), *we have*

$$I_{f,f_1}(N,h)=2N\sum_{1<\ell\le D}\ell^2R_\ell(f)R_\ell(f_1)\sum_{t|\ell}\frac{\mu(t)}{t^2}\Big\|\frac{h}{\ell/t}\Big\|+o(Nh);$$

whence, with an absolute constant,

$$I_{f,f_1}(N,h)\gg N\sum_{1<\ell\le\frac{D}{2}}\sum_{d\le\frac{D}{\ell}}\frac{g_1(\ell d)}{d}\sum_{q\le\frac{Q}{\ell}}\frac{g(\ell q)}{q}\sum_{t|\ell}\frac{\mu(t)}{t^2}\Big\|\frac{h}{\ell/t}\Big\|.$$

Furthermore, assuming also that $g(\ell q)\ge g(q)$ $\forall\ell\le Q,\ \forall q\le\frac{Q}{\ell}$, *we get the (absolute) lower bound*

$$I_{f,f_1}(N,h)\gg N\Big(\sum_{q\le\frac{Q}{D}}\frac{g(q)}{q}\Big)\sum_{1<\ell\le\frac{D}{2}}\sum_{d\le\frac{D}{\ell}}\frac{g_1(\ell d)}{d}\sum_{t|\ell}\frac{\mu(t)}{t^2}\Big\|\frac{h}{\ell/t}\Big\|.$$

Our "main" consequence is for the symmetry integral of d_k and for its Selberg integral, J_k, in the:

Corollary. *Fix* $k\ge 3$ integer. *Let* $N,h\in\mathbb{N}$ *give, say,* width $\theta=\theta_k$, $0<\theta_k<1/k$. *Then*

$$I_{d_k}(N,h)\gg_k NhL^{k+1},\quad J_k(N,h)\gg_k NhL^{k+1}.$$

We explicitly remark the coincidence that the width $<\frac{1}{k}$ is the range of h for which the J_k upper bound of the kind above (but it's a lower one !) is required, in order to get the (highly!) non-trivial bound (in [C4]) of ζ^{2k} integral-mean.

The paper is organized as follows:

- ⋄ in section 2 we state and prove our Lemma (on a "discrete mixed integral");
- ⋄ in section 3 we apply the Lemma (and an asymptotic formula) to prove our Theorem;
- ⋄ last section contains the proof of the Corollary, with some comments and remarks.

2. Statement and Proof of the Lemma.

Our Lemma, following, deals with "mixed symmetry integrals", defined above as:

$$I_{f,f_1}(N,h)=\int_N^{2N}\sum_{|n-x|\le h}\mathrm{sgn}(n-x)f(n)\sum_{|m-x|\le h}\mathrm{sgn}(m-x)f_1(m)dx,$$

where $f=g*\mathbf{1}$, $f_1=g_1*\mathbf{1}$; actually, the mean-square in the Lemma is a discrete one (a sum !), not an integral (like in the previous version of this paper). This is done in order to apply Lemma 2 in [C-S] (a kind of Large Sieve inequality, see its proof), dealing with Farey fractions (i.e., j/ℓ, r/t, with $(j,\ell)=1=(r,t)$, see the proof) and exploiting their "well-spaced" property (compare $(*)$ in the proof of the Lemma).

By the way, the first appearance of these ("mixed") integrals is in [C5], where (from the Cauchy-Schwarz inequality) they have non-trivial bounds, whenever one of the two "pure" (symmetry) integrals has one:

$$|I_{f,f_1}(N,h)| \le \int_N^{2N} \Big|\sum_{|n-x|\le h} \mathrm{sgn}(n-x)f(n)\Big|\Big|\sum_{|m-x|\le h} \mathrm{sgn}(m-x)f_1(m)\Big|dx \le \sqrt{I_f(N,h)}\sqrt{I_{f_1}(N,h)}.$$

Furthermore, we recall that the proof of the Lemma we use from our Acta Arithmetica paper relies solely on the Cauchy inequality. Hence, the present Lemma inherits the elementary character from that one.

In fact, it comes from the properties (see [C-S]) of the function

$$\chi_q(x) \stackrel{def}{=} \sum_{\substack{|n-x|\le h \\ n\equiv 0 (\mathrm{mod}\ q)}} \mathrm{sgn}(n-x),$$

entering the game, since (when g, g_1 have supports supp $(g_1) \subset [1,D]$, supp $(g) \subset [1,Q]$, here)

$$\sum_{x\sim N}\Big(\sum_{|n-x|\le h} \mathrm{sgn}(n-x) \sum_{\substack{q\le Q \\ q|n}} g(q)\Big)\Big(\sum_{|m-x|\le h} \mathrm{sgn}(m-x) \sum_{\substack{d\le D \\ d|m}} g_1(d)\Big) = \sum_{x\sim N}\sum_{q\le Q} g(q)\chi_q(x) \sum_{d\le D} g_1(d)\chi_d(x).$$

This DISCRETE MIXED INTEGRAL is linked to $I_{f,f_1}(N,h)$, see Thm. proof (§3). We treat the sum (and not the integral, as mistaken in v1, previous version !) of this double sum over these "character-like" functions. However, the Lemma still holds for $I_{f,f_1}(N,h)$ (as stated in v1, but will be proved within Thm. proof in §3).

We can (with Ramanujan coefficients $R_\ell(f)$ defined in the Thm. above) state and show our

LEMMA. *Let* $N, h \in \mathbb{N}$ *with* $h \to \infty$ *and* $h = o(N)$ *when* $N \to \infty$. ASSUME $g, g_1 : \mathbb{N} \to \mathbb{C}$ *with* $g_1(d) = 0$ $\forall d > D$ *and* $g(q) = 0$ $\forall q > Q$, *where* $1 < D \le Q \ll N$. THEN

$$\sum_{x\sim N}\Big(\sum_{q\le Q} g(q)\chi_q(x) \sum_{d\le D} g_1(d)\chi_d(x)\Big) = 2N \sum_{1<\ell\le D} \ell^2 R_\ell(f)R_\ell(f_1) \sum_{t|\ell} \frac{\mu(t)}{t^2} \left\|\frac{h}{\ell/t}\right\| +$$

$$+\mathcal{O}\left(DQL\sqrt{\sum_{1<t\le 2h} t^2|R_t(f_1)|^2 + h\sum_{2h<t\le D} t|R_t(f_1)|^2}\sqrt{\sum_{1<\ell\le 2h} \ell^2|R_\ell(f)|^2 + h\sum_{2h<\ell\le Q} \ell|R_\ell(f)|^2}\right).$$

PROOF. Abbreviate $n \equiv a(q)$ for $n \equiv a(\mathrm{mod}\ q)$ and, from ADDITIVE CHARACTERS ORTHOGONALITY [V],

$$\chi_q(x) = \sum_{\substack{|r|\le h \\ r\equiv -x(q)}} \mathrm{sgn}(r) = \sum_{j<q} c_{j,q}e_q(jx)$$

get the FOURIER COEFFICIENTS (of previous finite Fourier expansion), see [C-S],

$$c_{j,q} := \frac{1}{q}\sum_{|r|\le h} \mathrm{sgn}(r)e_q(rj) \quad \text{SATISFYING} \quad c_{dj',dq'} = \frac{1}{d}c_{j',q'}, \forall d, j', q' \in \mathbb{N}, \quad \text{WHENCE}$$

$$\chi_q(x) = \sum_{\substack{\ell|q \\ \ell>1}} \frac{\ell}{q} \sideset{}{^*}\sum_{j\le\ell} c_{j,\ell}e_\ell(jx), \quad \text{WITH} \quad \sum_{j<q} |c_{j,q}|^2 = 2\left\|\frac{h}{q}\right\|, \qquad \sideset{}{^*}\sum_{j<\ell} |c_{j,\ell}|^2 = 2\sum_{t|\ell} \frac{\mu(t)}{t^2}\left\|\frac{ht}{\ell}\right\|.$$

By the way, this last relation highlights: THE SUM ABOVE, performed over $t|\ell$, IS NON-NEGATIVE. Then

$$\sum_{x\sim N}\sum_{d\le D}\sum_{q\le Q} g_1(d)g(q)\chi_d(x)\chi_q(x) =$$

$$= \sum_{1<t\le D} \left(\sum_{d'\le \frac{D}{t}} \frac{g_1(td')}{d'} \right) \sum_{1<\ell\le Q} \left(\sum_{q'\le \frac{Q}{\ell}} \frac{g(\ell q')}{q'} \right) \sum_{r(t)}^{*} \overline{c_{r,t}} \sum_{j(\ell)}^{*} c_{j,\ell} \sum_{x\sim N} e(\alpha x)$$

(APPLY PREVIOUS PROPERTIES of χ_q EXPANSION), with, say, $\alpha := \frac{j}{\ell} - \frac{r}{t}$; apply Lemma 2 [C-S], SINCE

$$(*) \qquad \|\alpha\| \ne 0 \;\Rightarrow\; \sum_{x\sim N} e(\alpha x) = e(\alpha/2) \frac{e(2N\alpha) - e(N\alpha)}{2i \sin \pi\alpha} \ll \frac{1}{\|\alpha\|}$$

together with $\dfrac{j}{\ell} \ne \dfrac{r}{t} \Rightarrow \left\| \dfrac{j}{\ell} - \dfrac{r}{t} \right\| \ge \dfrac{1}{\ell t} \gg \dfrac{1}{DQ} \quad \forall t \le D \; \forall \ell \le Q$ (RECALL THEY'RE FAREY FRACTIONS) GIVE $\frac{1}{DQ}$ WELL-SPACED (FAREY) FRACTIONS AND (ISOLATING $\frac{j}{\ell} = \frac{r}{t} \Rightarrow \ell = t$, I.E. THE "DIAGONAL")

$$\sum_{x\sim N} \Big(\sum_{q\le Q} g(q)\chi_q(x) \sum_{d\le D} g_1(d)\chi_d(x) \Big) = \sum_{1<\ell\le D} \left(\sum_{d\le \frac{D}{\ell}} \frac{g_1(\ell d)}{d} \right) \left(\sum_{q\le \frac{Q}{\ell}} \frac{g(\ell q)}{q} \right) \left(\sum_{j<\ell}^{*} |c_{j,\ell}|^2 \right) N +$$

$$+\mathcal{O}\left(DQL \sqrt{ \sum_{1<t\le D} \left| \sum_{d\le \frac{D}{t}} \frac{g_1(td)}{d} \right|^2 \sum_{r<t}^{*} |c_{r,t}|^2 } \sqrt{ \sum_{1<\ell\le Q} \left| \sum_{q\le \frac{Q}{\ell}} \frac{g(\ell q)}{q} \right|^2 \sum_{j<\ell}^{*} |c_{j,\ell}|^2 } \right).$$

From the above property of χ_q we MAY use : $\sum_{j<\ell}^{*} |c_{j,\ell}|^2 \ll \sum_{j<\ell} |c_{j,\ell}|^2 \ll \min(1, \frac{h}{\ell})$ to get

$$\sum_{1<t\le D} \left| \sum_{d\le \frac{D}{t}} \frac{g_1(td)}{d} \right|^2 \sum_{r<t}^{*} |c_{r,t}|^2 \ll \sum_{1<t\le 2h} \left| \sum_{d\le \frac{D}{t}} \frac{g_1(td)}{d} \right|^2 + h \sum_{2h<t\le D} \frac{1}{t} \left| \sum_{d\le \frac{D}{t}} \frac{g_1(td)}{d} \right|^2$$

and

$$\sum_{1<\ell\le Q} \left| \sum_{q\le \frac{Q}{\ell}} \frac{g(\ell q)}{q} \right|^2 \sum_{j<\ell}^{*} |c_{j,\ell}|^2 \ll \sum_{1<\ell\le 2h} \left| \sum_{q\le \frac{Q}{\ell}} \frac{g(\ell q)}{q} \right|^2 + h \sum_{2h<\ell\le Q} \frac{1}{\ell} \left| \sum_{q\le \frac{Q}{\ell}} \frac{g(\ell q)}{q} \right|^2,$$

WHENCE the (remainders, i.e. the) OFF-DIAGONAL TERMS ARE

$$\ll DQL \sqrt{ \sum_{1<t\le 2h} \left| \sum_{d\le \frac{D}{t}} \frac{g_1(td)}{d} \right|^2 + h \sum_{2h<t\le D} \frac{1}{t} \left| \sum_{d\le \frac{D}{t}} \frac{g_1(td)}{d} \right|^2 } \times$$

$$\times \sqrt{ \sum_{1<\ell\le 2h} \left| \sum_{q\le \frac{Q}{\ell}} \frac{g(\ell q)}{q} \right|^2 + h \sum_{2h<\ell\le Q} \frac{1}{\ell} \left| \sum_{q\le \frac{Q}{\ell}} \frac{g(\ell q)}{q} \right|^2 };$$

and, using the definition of RAMANUJAN COEFFICIENTS (see Thm.), we get the desired estimate. □

3. Proof of the Theorem.

PROOF. First of all, we link $I_{f,f_1}(N,h)$ with the discrete mixed integral of the Lemma: define

$$S_f^{\pm}(x) \overset{def}{=} \sum_{|n-x|\le h} f(n)\mathrm{sgn}(n-x),$$

the SYMMETRY SUM of the (real) arithmetic function f. Obviously, $f \lll 1 \Rightarrow S_f^{\pm} \lll h$. Then

$$I_{f,f_1}(N,h) = \int_N^{2N} \Big(S_f^{\pm}([x]) - f([x]) + f([x]-h)\Big)\Big(S_{f_1}^{\pm}([x]) - f_1([x]) + f_1([x]-h)\Big)\,dx =$$

$$= \sum_{N \le x < 2N} \Big(S_f^{\pm}(x) - f(x) + f(x-h)\Big)\Big(S_{f_1}^{\pm}(x) - f_1(x) + f_1(x-h)\Big) =$$

$$= \sum_{x \sim N} \Big(S_f^{\pm}(x) - f(x) + f(x-h)\Big)\Big(S_{f_1}^{\pm}(x) - f_1(x) + f_1(x-h)\Big) + \mathcal{O}_{\varepsilon}\left(N^{\varepsilon}h^2\right).$$

Here the x is intended both real (in $\int$) and natural (in $\sum$); but the integral doesn't see the $x \in \mathbb{N}$. Due to the hypothesis $\theta < 1/2$ ($\Rightarrow \theta < 1$), this error term is $o(Nh)$. Now, this sum is

$$\sum_{x \sim N} \left(\sum_{q \le Q} g(q)\chi'_q(x) \sum_{d \le D} g_1(d)\chi'_d(x) \right),$$

which is not treated in the Lemma, because here ($x \in \mathbb{N}$ and $\mathbf{1}_{\wp} = 1$ if $\wp$ is true, 0 otherwise) :

$$\chi'_q(x) \stackrel{def}{=} \sideset{}{'}\sum_{\substack{|n-x| \le h \\ n \equiv 0 (\bmod q)}} \mathrm{sgn}(n-x) = \sum_{\substack{x < n \le x+h \\ n \equiv 0 (\bmod q)}} 1 - \sum_{\substack{x-h < n \le x \\ n \equiv 0 (\bmod q)}} 1 = \chi_q(x) - \mathbf{1}_{q|x} + \mathbf{1}_{q|x-h},$$

i.e. the dash means that $n = x$ is counted with "$-$" sign and $n = x - h$ is not counted. If we consider

$$\sum_{q \le Q} g(q)\chi'_q(x) - \sum_{q \le Q} g(q)\chi_q(x) = \sum_{q|x-h, q \le Q} g(q) - \sum_{q|x, q \le Q} g(q) \lll d(x-h) + d(x) \lll 1,$$

we have that the present mean-square and the one in the Lemma differ by $\lll Nh$. This is not negligible. However, the same proof of the Lemma, applied to

$$\chi'_q(x) \text{ INSTEAD OF } \chi_q(x), \text{ WITH } c'_{j,q} \text{ INSTEAD OF } c_{j,q},$$

i.e. giving again the (finite) Fourier expansion, but with, say, the FOURIER COEFFICIENTS

$$c'_{j,q} := \frac{1}{q} \sideset{}{'}\sum_{|r| \le h} \mathrm{sgn}(r) e_q(rj)$$

(the dash takes $r = 0$ with "$-$" and doesn't count $r = -h$), we may repeat Lemma proof verbatim to

$$\chi'_q(x) = \sum_{\substack{\ell | q \\ \ell > 1}} \frac{\ell}{q} \sideset{}{^*}\sum_{j \le \ell} c'_{j,\ell} e_\ell(jx), \text{ WITH } \sum_{j<q} \left|c'_{j,q}\right|^2 = 2\left\|\frac{h}{q}\right\|, \quad \sideset{}{^*}\sum_{j<\ell} \left|c'_{j,\ell}\right|^2 = 2\sum_{t|\ell} \frac{\mu(t)}{t^2}\left\|\frac{ht}{\ell}\right\|,$$

getting (see the above; by the way, this recovers the Lemma in version v1)

$$I_{f,f_1}(N,h) = 2N \sum_{1 < \ell \le D} \ell^2 R_\ell(f) R_\ell(f_1) \sum_{t|\ell} \frac{\mu(t)}{t^2}\left\|\frac{h}{\ell/t}\right\| + o(Nh) +$$

$$+ \mathcal{O}\left(DQL \sqrt{\sum_{1<t\le 2h} t^2|R_t(f_1)|^2 + h \sum_{2h<t\le D} t|R_t(f_1)|^2} \sqrt{\sum_{1<\ell\le 2h} \ell^2|R_\ell(f)|^2 + h \sum_{2h<\ell\le Q} \ell|R_\ell(f)|^2} \right).$$

This holds for $\theta < 1/2$ (as we use it for the Thm.), but is true in the Lemma hypotheses, joining $\theta < 1$. An immediate application of this gives Thm. equation, using $R_\ell(f), R_\ell(f_1) \lll \frac{1}{\ell}$ above, since $\delta + \lambda < 1$.

Then, due to : $g_1, g \geq 1$,

$$2N \sum_{1<\ell\leq D} \ell^2 R_\ell(f) R_\ell(f_1) \sum_{t|\ell} \frac{\mu(t)}{t^2} \Big\| \frac{h}{\ell/t} \Big\| = 2N \sum_{1<\ell\leq D} \sum_{d\leq \frac{D}{\ell}} \frac{g_1(\ell d)}{d} \sum_{q\leq \frac{Q}{\ell}} \frac{g(\ell q)}{q} \sum_{t|\ell} \frac{\mu(t)}{t^2} \Big\| \frac{h}{\ell/t} \Big\| \gg$$

$$\gg N \left(\sum_{q\leq \frac{Q}{D}} \frac{1}{q} \right) \sum_{1<\ell\leq D} \sum_{t|\ell} \frac{\mu(t)}{t^2} \Big\| \frac{h}{\ell/t} \Big\| \gg N \left(\sum_{q\leq \frac{Q}{D}} \frac{1}{q} \right) \sum_{t<\frac{D}{2h}} \frac{\mu(t)}{t^2} \left(\sum_{1<n\leq 2h} \Big\| \frac{h}{n} \Big\| + h \sum_{2h<n\leq \frac{D}{t}} \frac{1}{n} \right) \gg$$

$$\gg Nh \log \frac{Q}{D} \sum_{t<\frac{D}{2h}} \frac{\mu(t)}{t^2} \sum_{2h<n\leq \frac{D}{t}} \frac{1}{n} \gg Nh \log \frac{Q}{D} \sum_{2h<n\leq D} \frac{1}{n} \sum_{t\leq \frac{D}{n}} \frac{\mu(t)}{t^2} \gg Nh \log \frac{Q}{D} \log \frac{D}{2h} \gg NhL^2,$$

where we used $\log \frac{Q}{D} \gg L$ (from $\delta < \lambda$) and $\log \frac{D}{2h} \gg L$ (from $\delta > \theta$) in the well-known (see, esp., [T]):

$$\sum_{t\leq T} \frac{\mu(t)}{t^2} = \frac{1}{\zeta(2)} + \mathcal{O}\left(\frac{1}{T}\right).$$

Hence, the main term is $\gg Nh$, whence $o(Nh)$ can be neglected (with an absolute constant in the $\gg$):

$$2N \sum_{1<\ell\leq D} \ell^2 R_\ell(f) R_\ell(f_1) \sum_{t|\ell} \frac{\mu(t)}{t^2} \Big\| \frac{h}{\ell/t} \Big\| \gg N \sum_{1<\ell\leq \frac{D}{2}} \sum_{d\leq \frac{D}{\ell}} \frac{g_1(\ell d)}{d} \sum_{q\leq \frac{Q}{\ell}} \frac{g(\ell q)}{q} \sum_{t|\ell} \frac{\mu(t)}{t^2} \Big\| \frac{h}{\ell/t} \Big\|.$$

Recall, always in our calculations, that $g_1, g \geq 1$ and the sum over $t|\ell$ is ≥ 0, see Lemma proof.
For the same reasons, the additional hypothesis on g gives at once $I_{f,f_1}(N,h)$ lower bound:

$$N \sum_{1<\ell\leq \frac{D}{2}} \sum_{d\leq \frac{D}{\ell}} \frac{g_1(\ell d)}{d} \sum_{q\leq \frac{Q}{\ell}} \frac{g(\ell q)}{q} \sum_{t|\ell} \frac{\mu(t)}{t^2} \Big\| \frac{h}{\ell/t} \Big\| \gg N\Big(\sum_{q\leq \frac{Q}{D}} \frac{g(q)}{q} \Big) \sum_{1<\ell\leq \frac{D}{2}} \sum_{d\leq \frac{D}{\ell}} \frac{g_1(\ell d)}{d} \sum_{t|\ell} \frac{\mu(t)}{t^2} \Big\| \frac{h}{\ell/t} \Big\|. \ \square$$

4. Proof of the Corollary. Remarks and comments.

PROOF. We recall the definition of SYMMETRY SUM for f (see Thm. proof)

$$S_f^{\pm}(x) = \sum_{|n-x|\leq h} \mathrm{sgn}(n-x) f(n)$$

and, in particular, for $f = d_k$ (the $k-$divisor function, generated by ζ^k), we write

$$S_k^{\pm}(x) \stackrel{def}{=} \sum_{|n-x|\leq h} d_k(n)\mathrm{sgn}(n-x) = \sum_{\substack{d_1 \\ |d_1\cdots d_k - x|\leq h}} \cdots \sum_{d_k} \mathrm{sgn}(d_1 \cdots d_k - x)$$

where, considering that (in our symmetry sum)
here $x \geq N \Rightarrow x-h \geq N-h \Rightarrow$ AT LEAST ONE OF $d_1, \ldots, d_k$ HAS TO BE $d_j \geq (N-h)^{1/k}$, do the following:

LET'S CALL Σ_0 THE PART OF $S_k^{\pm}$ IN WHICH $d_1 \geq (N-h)^{1/k}$; REMAINS $S_k^{\pm} - \Sigma_0$, IN WHICH $d_2 \geq (N-h)^{1/k}$,

LET'S CALL IT Σ_1; REMAINS $S_k^{\pm} - \Sigma_0 - \Sigma_1$, IN WHICH $d_3 \geq (N-h)^{1/k}$, SAY Σ_2, AND SO ON.

Since in $S_k^{\pm}$ AT LEAST ONE of $d_1,\dots,d_k$ HAS TO BE $\ge (N-h)^{1/k}$, WE GET

$$S_k^{\pm}(x)=\sum_{q\le \frac{x+h}{(N-h)^{1/k}}} d_{k-1}(q)\sum_{\substack{\left|m-\frac{x}{q}\right|\le\frac{h}{q}\\ m\ge (N-h)^{1/k}}}\operatorname{sgn}\left(m-\frac{x}{q}\right)+\sum_{q\le \frac{x+h}{(N-h)^{1/k}}} d_{k-1}^{(1)}(q)\sum_{\substack{\left|m-\frac{x}{q}\right|\le\frac{h}{q}\\ m\ge (N-h)^{1/k}}}\operatorname{sgn}\left(m-\frac{x}{q}\right)+$$

$$+\cdots+\sum_{q\le \frac{x+h}{(N-h)^{1/k}}} d_{k-1}^{(k-1)}(q)\sum_{\substack{\left|m-\frac{x}{q}\right|\le\frac{h}{q}\\ m\ge (N-h)^{1/k}}}\operatorname{sgn}\left(m-\frac{x}{q}\right),$$

where $d_{k-1}^{(0)}(q):=d_{k-1}(q)$ HAS RESTRICTIONS ON 0 FACTORS,

$$d_{k-1}^{(1)}(q):=\sum_{d_1}\cdots\sum_{\substack{d_{k-1}\\ d_1\cdots d_{k-1}=q\\ d_1<(N-h)^{1/k}}}1$$

HAS ON 1 FACTOR, and $\forall j\le k-1$,

$$d_{k-1}^{(j)}(q):=\sum_{d_1}\cdots\sum_{\substack{d_{k-1}\\ d_1\cdots d_{k-1}=q\\ d_1,\dots,d_j<(N-h)^{1/k}}}1$$

HAS j FACTORS WITH RESTRICTIONS (WHICH ARE INDEPENDENT OF x !).
Hence, CALLING $g(q):=\sum_{j=0}^{k-1}d_{k-1}^{(j)}(q)$, (DEPENDS on $(N-h)^{1/k}$, TOO), GET $1\le g(q)\le kd_{k-1}(q)\lll_k 1$.
We OBTAIN immediately that the symmetry sum $S_k^{\pm}(x)$ equals (we'll ignore the constants $k-$dependence)

$$\sum_{q\le \frac{x+h}{(N-h)^{1/k}}} g(q)\sum_{\substack{\left|m-\frac{x}{q}\right|\le\frac{h}{q}\\ m\ge (N-h)^{1/k}}}\operatorname{sgn}\left(m-\frac{x}{q}\right)=\sum_{q\le \frac{x-h}{(N-h)^{1/k}}} g(q)\chi_q(x)+\mathcal{O}_\varepsilon\left(\sum_{\frac{x-h}{(N-h)^{1/k}}<q\le\frac{x+h}{(N-h)^{1/k}}}\ \sum_{\left|m-\frac{x}{q}\right|\le\frac{h}{q}}x^\varepsilon\right).$$

In these remainders, $q>\frac{x-h}{(N-h)^{1/k}}\gg N^{1-1/k}$ (as $N\le x\le 2N$ in the integral) gives (from our hypotheses $k>2$ and $\theta<1/k$) that $h=o(q)$, whence the interval $[\frac{x-h}{q},\frac{x+h}{q}]$ contains (at most) one integer m (the $m-$sum is "SPORADIC") and this, in turn, implies that the remainders are :

$$\lll \sum_{\frac{x-h}{(N-h)^{1/k}}<q\le\frac{x+h}{(N-h)^{1/k}}}\ \sum_{\left|m-\frac{x}{q}\right|\le\frac{h}{q}}1\lll 1,$$

since the $q-$sum, too, contains at most one integer (SPORADICITY from: $\theta<1/k$).

However, from Cauchy-Schwarz inequality, this contributes $\lll Nh$, giving "interference" with the lower bound (of the same order of magnitude, say DIAGONAL-like). We need a slight improvement on this bound for the remainder; this is done estimating its mean-square (recall, we're to find a lower bound for its $N\le x\le 2N$ integral!): bounding $S_k^{\pm}(x)\lll h$ (trivially), the contribute in the integral due to these remainders becomes (apply the SPORADICITY ARGUMENT to the inner $q-$sum)

$$\lll h\int_N^{2N}\sum_{\frac{x-h}{(N-h)^{1/k}}<q\le\frac{x+h}{(N-h)^{1/k}}}\ \sum_{\left|m-\frac{x}{q}\right|\le\frac{h}{q}}1\,dx\lll h\max_{\frac{N-h}{(N-h)^{1/k}}<q\le\frac{2N+h}{(N-h)^{1/k}}}\int_N^{2N}\sum_{\left|m-\frac{x}{q}\right|\le\frac{h}{q}}1\,dx$$

$$\lll h^2\max_{\frac{N-h}{(N-h)^{1/k}}<q\le\frac{2N+h}{(N-h)^{1/k}}}\ \sum_{\frac{N-h}{q}\le m\le\frac{2N+h}{q}}1\lll Nh\Big(\frac{h}{N^{1-1/k}}\Big),$$

which is $o(Nh)$, since (recall: $k>2$) we have WIDTH $\theta<1/k<1-1/k$.

Now on, we will ignore all of the $o(Nh)$ contributes to our integrals.
Writing "$\sim$" to mean we're leaving (such) negligible remainders, we are left with

$$S_k^{\pm}(x)\sim\sum_{q\le\frac{x-h}{(N-h)^{1/k}}}g(q)\chi_q(x)=\sum_{q\le Q}g(q)\chi_q(x)+\sum_{Q<q\le\frac{x-h}{(N-h)^{1/k}}}g(q)\chi_q(x),$$

where we set $Q:=(N-h)^{1-1/k}$; whence, we are enabled to say that $\lambda:=1-1/k$ is the LEVEL.

In fact, the same arguments of our Lemma give the same estimates for non-diagonal terms in the case we have the further limitation $q\le\frac{x-h}{(N-h)^{1/k}}$, which depends on x, since $(*)$, in the Proof of the Lemma, holds whatever limitations hold on the summation interval; also, we get from the second sum a positive (better, non-negative) contribution, for our symmetry integral (say, "on the diagonal").

Hence, we will ignore the second sum (from a positivity argument, to be applied soon again).

Finally, we may also ignore (see the remarks, following soon after) the parts inside g having limitations on the factors (say, consider $g(q):=d_{k-1}(q)$, here). In all, we are left, after applying the Theorem (with δ auxiliary level, $\theta_k<\delta<1/k$, $D:=[N^\delta]$ and $g_1=\mathbf{1}$; also, $d_{k-1}(\ell q)\ge d_{k-1}(q)$), to saying that (1), together with the bound ([C2], compare [C-S]) $I_{g_1*\mathbf{1}}(N,h)\ll NhL^3$ (thanks to exponent $3<k+1$, $\forall k>2$), gives

$$I_{d_k}(N,h)\gg_k N\left(\sum_{q\le\frac{Q}{D}}\frac{d_{k-1}(q)}{q}\right)\sum_{1<\ell\le\frac{D}{2}}\log\frac{D}{\ell}\sum_{t|\ell}\frac{\mu(t)}{t^2}\left\|\frac{h}{\ell/t}\right\|\gg_k$$

$$\gg_k N\left(\sum_{q\le\frac{Q}{D}}\frac{d_{k-1}(q)}{q}\right)\left(\sum_{1<n\le 2h}\left\|\frac{h}{n}\right\|\sum_{t<\frac{D}{2h}}\frac{\mu(t)}{t^2}\log\frac{D}{nt}+h\sum_{2h<n\le D}\frac{1}{n}\sum_{t\le\frac{D}{n}}\frac{\mu(t)}{t^2}\log\frac{D}{nt}\right)\gg_k$$

$$\gg_k Nh\left(\sum_{q\le\frac{Q}{D}}\frac{d_{k-1}(q)}{q}\right)\sum_{2h<n\le D}\frac{1}{n}\sum_{t\le\frac{D}{n}}\frac{\mu(t)}{t^2}\log\frac{D}{nt}\gg_k NhL^2\left(\sum_{q\le\frac{Q}{D}}\frac{d_{k-1}(q)}{q}\right),$$

as in Theorem proof ($0<\theta<\delta$), having used [T]

$$\sum_{2h<n\le D}\frac{1}{n}\sum_{t\le\frac{D}{n}}\frac{\mu(t)}{t^2}\log\frac{D}{nt}=\sum_{2h<n\le D}\frac{1}{n}\log\frac{D}{n}\left(\frac{1}{\zeta(2)}+\mathcal{O}\left(\frac{n}{D}\right)\right)-\sum_{2h<n\le D}\frac{1}{n}\left(\sum_{t=1}^{\infty}\frac{\mu(t)\log t}{t^2}+\mathcal{O}\left(\frac{n}{D}L\right)\right),$$

together with partial summation [T] (compare [C-S] Corollary 1 calculations, p.199 on); hence

$$I_{d_k}(N,h)\gg_k NhL^{k+1},$$

this last inequality coming from partial summation (see [D] or [T]) and (Lemma 1.1.2 of) Ch.1 of [L], as

$$\sum_{n\le x}\frac{d_{k-1}(n)}{n}=\sum_{\substack{n_1,\ \dots,\ n_{k-1}\\ n_1\cdots n_{k-1}\le x}}\frac{1}{n_1\cdots n_{k-1}}\gg_k(\log x)^{k-1}.$$

As regards the lower bound for Selberg integral J_k, we apply previous connection, with (in [C4] details)

$$M_k(x,h):=hP_{k-1}(\log x),$$

where $P_{k-1}(\log x)$ is a $k-1$ degree polynomial in $\log x$, whence

$$M_k(x,h)-M_k(x-h,h)=hM_k'(x-\alpha h,h)\lll h^2/N,\qquad\forall x\in[N,2N]$$

(from mean-value theorem, with $M'_k(x,h) := \frac{d}{dx} M_k(x,h)$, $0 < \alpha < 1$), whence ("$\gg$" leaves $o(Nh)$, here)

$$J_k(N,h) \gg I_{d_k}(N,h). \quad \square$$

We remark we "wasted", in our (previous, version 1) lower bounds, "*many*" terms in our previous analysis.

In fact, we felt that the limitation $k \ge 5$ was immaterial.

Actually, the real improvement comes from the (previously) neglected terms of the Theorem, where the Möbius function rendered more cumbersome our estimates (simplified by the hypothesis $g(\ell q) \ge g(q)$, here).

Once again, we are postponing other eventual, further improvements to a future, forthcoming paper.

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