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Number Theory

G. Coppola – M.B.S. Laporta

ON THE REPRESENTATION OF EVEN INTEGERS AS SUM OF TWO ALMOST EQUAL PRIMES

Abstract. In this paper we generalize the Chudakov - van der Corput - Estermann Theorem on the exceptional set in the binary Goldbach problem to a result on the same problem with "almost equal" primes. Actually, we prove that the equation $p_1 + p_2 = 2n$ is satisfied by almost all $2n \in [N, 2N]$ when the primes p_1 and p_2 lie in the interval $[n - U, n + U]$, with $U = n^{5/8+\varepsilon}$. Furthermore, we explicitly estimate the number of representations of these $2n$ as sums of such primes.

1. Introduction

In the binary Goldbach problem, from the Chudakov-van der Corput- Estermann theorem [11, ch.3] it follows that the number $E(N)$ of even numbers n not exceeding N for which n is not the sum of two primes satisfies $E(N) \ll NL^{-A}$, where $L = \log N$ and $A > 0$ is an absolute constant.

Vinogradov's three primes theorem [12] states that every sufficiently large odd integer N can be represented as $N = p_1 + p_2 + p_3$, with primes p_i, $i = 1, 2, 3$. Several authors proved that this theorem still holds under the further restriction $|p_i - N/3| \ll N^{\theta+\varepsilon}$ ($\theta < 1$), $i = 1, 2, 3$ (see [2], [8], [1], [9], where respectively $\theta = 63/64$, $160/183$, $2/3$, $91/96$).

Assuming the Generalized Riemann Hypothesis (GRH), Wolke [13] established the same result with $|p_i - N/3| \leq N^{1/2}(\log N)^{7+\varepsilon}$, $i = 1, 2, 3$.

Subsequently, improving the previous unconditional results, Zhan Tao [14] obtained the theorem for $\theta = 5/8$. Moreover, his method also yields an asymptotic formula for the number $R(N)$ of representations. Using the circle method, the problem in [14] reduces to obtaining nontrivial estimates for exponential sums over primes in short intervals. In some cases, the estimates depend on density estimates in short intervals. In other cases, Heath-Brown's identity and some analytic techniques based on mean values of the Dirichlet L-functions are applied.

In this paper, using Zhan Tao's result, we show how one could obtain an analogous result for the corresponding binary problem

$$2n = p_1 + p_2, \quad n - U \le p_1, p_2 \le n + U. \tag{1.1}$$

More precisely, let

$$R(2n) = R(2n, U) = \sum_{\substack{h+k=2n \\ n-U<h,k\le n+U}} \Lambda(h)\Lambda(k),$$

$$\mathfrak{S}(2n) = 2 \prod_{\substack{p|n \\ p>2}} \left(\frac{p-1}{p-2}\right) \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right),$$

where Λ is von Mangoldt's function. Then we have

THEOREM 1. *Let $\varepsilon > 0$ and $A > 0$ be arbitrary constants and let $N^{5/8+\varepsilon} \le U \le N$. Then*

$$\sum_{N\le 2n\le 2N} |R(2n) - 2U\mathfrak{S}(2n)|^2 \ll_{\varepsilon,A} NU^2L^{-A}.$$

Theorem 1 implies

COROLLARY. *Let $\varepsilon > 0$ and $A > 0$ be arbitrary constants . Then for all $2n \in [N, 2N]$ but $O(NL^{-A})$ exceptions, the equation (1.1) is solvable for $U = n^{5/8+\varepsilon}$ and we have*

$$R(2n) = 2U\mathfrak{S}(2n) + O(UL^{-A}).$$

We remark that further refinements of Zhan Tao's result have been obtained by Jia [3]-[5], with $U = n^{7/12+\varepsilon}$ in [5]. However, he uses sieve methods, which lead only to a lower bound for the number of representations. Jia deduces his result by establishing that for an even integer n such that $2P < n \le 2P + U$ with $O(U \log^{-2} P)$ exceptions, one has

$$|\{(p_1, p_2) : n = p_1 + p_2,\ P < p_1 \le P+U,\ P-U < p_2 \le P+U\}| \gg \mathfrak{S}(n)\frac{U}{\log^2 P}, \tag{1.2}$$

where P is a sufficiently large number, ε is a sufficiently small positive number, and $U = P^{7/12+40\varepsilon}$. Recently, Mikawa [7] has independently established the same result of [5] without appealing to a theorem of the form (1.2). Then, adopting the approach of Mikawa, the second author [6] has obtained the following generalization of the results concerning the binary problem.

Let us consider the linear equation

$$n = Ap_1 + Bp_2, \quad n/2 - U \le Ap_1, Bp_2 \le n/2 + U, \tag{1.3}$$

where A and B are any positive integers such that $(A, B) = 1$ and $n \in \mathcal{A} = \{n : (AB, n) = 1,\ n \equiv A + B (\bmod 2)\}$. Then we have

THEOREM 2. *Let ε and E be arbitrary positive constants, and N be a sufficiently large integer. Then for every $n \in \mathcal{A}_N = [N, 2N] \cap \mathcal{A}$ with $O(N \log^{-E} N)$ exceptions, the equation* (1.3) *is solvable for $U = n^{7/12+\varepsilon}$, for $1 \le A, B \ll \log^c N$ and $(A, B) = 1$, where $c = c(E) > 0$.*

2. Notation

A - an arbitrary positive constant,

B, C, D - positive constants depending on A,

ε - an arbitrarily small positive constant,

V,Y - positive integers such that $N^{7/12+\varepsilon} \le Y \le V$,

N,U - positive integers such that $N^{5/8+\varepsilon} \le U \le N$,

$H = UL^{-B}$,

$T \sim \frac{N}{4H}$,

$\omega = N^{-1}U^2L^{-D}$,

$P = L^C$,

$n_1, n_2, \ldots, n_T$ - integers such that $N/2 \le n_1 < n_2 < \ldots < n_T \le N$,

$$S(\alpha; V, Y) = \sum_{V-Y<h\le V} \Lambda(h)e(h\alpha),$$

$$S_n(\alpha) = S(\alpha; n+U, 2U), \quad T_n(\eta) = \sum_{n-U<h\le n+U} e(h\eta).$$

$$I_{q,a} = \left\{\frac{a}{q} + \eta, \eta \in \xi_q\right\}, \text{ with } \xi_q = \left(-\frac{L^D}{U}, \frac{L^D}{U}\right).$$

$$\mathfrak{M} = \bigcup_{q\le P} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} I_{q,a}, \quad \text{and} \quad \mathfrak{m} = \left[\frac{1}{\omega}, 1 + \frac{1}{\omega}\right] \setminus \mathfrak{M}.$$

$$\sum_{a=1}^{q}{}^{*} = \sum_{\substack{a=1\\(a,q)=1}}^{q}, \qquad \delta_\chi = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \ne \chi_0, \end{cases} \qquad \|\beta\| = \min_{n\in\mathbb{Z}} |\beta - n|.$$

$$\tau(\chi) = \sum_{a=1}^{q}{}^{*} \chi(a)e\left(\frac{a}{q}\right), \text{ Gauss' sum,}$$

$$\psi(x,\chi)=\sum_{n\le x}\Lambda(n)\chi(n).$$

3. Outline of the method

We have that

$$R(2n)=\int_0^1 S_n(\alpha)^2 e(-2n\alpha)d\alpha.$$

Then we have

$$\sum_{N\le 2n\le 2N}|R(2n)-2U\mathfrak{S}(2n)|^2\ll\sum_{\mathfrak{m}}+\sum_{\mathfrak{M}},$$

where

$$\sum_{\mathfrak{m}}=\sum_{N\le 2n\le 2N}\left|\int_{\mathfrak{m}}S_n(\alpha)^2 e(-2n\alpha)d\alpha\right|^2,$$

$$\sum_{\mathfrak{M}}=\sum_{N\le 2n\le 2N}\left|\int_{\mathfrak{M}}S_n(\alpha)^2 e(-2n\alpha)d\alpha-2U\mathfrak{S}(2n)\right|^2.$$

In order to prove Theorem 1 it suffices to show that

$$\sum_{\mathfrak{M}}\ll NU^2L^{-A}, \tag{3.1}$$

$$\sum_{\mathfrak{m}}\ll NU^2L^{-A}. \tag{3.2}$$

4. The major arcs

Let $\alpha\in I_{q,a}$. Then we have

$$S_n(\alpha)=\sum_{n-U<h\le n+U}\Lambda(h)e(h\alpha)=$$

$$\sum_{\substack{n-U<h\le n+U\\(h,q)=1}}\Lambda(h)e\left(\frac{ah}{q}\right)e(h\eta)+O(L^2)=$$

$$\sum_{b=1}^{q}{}^{*}e\left(\frac{ab}{q}\right)\sum_{\substack{n-U<h\le n+U\\h\equiv b\ (\mathrm{mod}\ q)}}\Lambda(h)e(h\eta)+O(L^2)=$$

$$\frac{1}{\varphi(q)}\sum_{\chi}\sum_{n-U<h\le n+U}\Lambda(h)\chi(h)e(h\eta)\sum_{b=1}^{q}{}^{*}\overline{\chi}(b)e\left(\frac{ab}{q}\right)+O(L^2)=$$

$$\frac{1}{\varphi(q)}\sum_{\chi}\chi(a)\tau(\overline{\chi})W_n(\chi,\eta)+\frac{\mu(q)}{\varphi(q)}T_n(\eta)+O(L^2)$$

where

$$W_n(\chi,\eta)=\sum_{n-U<h\le n+U}\Lambda(h)\chi(h)e(h\eta)-\delta_\chi T_n(\eta).$$

Now we use the following short intervals version of the Siegel-Walfisz theorem (see [10]). If $V^{7/12+\varepsilon}\le Y\le V$ and $q\le P=L^C$, then

$$\psi(V,\chi)-\psi(V-Y,\chi)=\delta_\chi Y+O_{\varepsilon,C,C_1}(YL^{-C_1}),$$

for every χ (mod q) and $C_1>0$.

If $U\ge n^{5/8+\varepsilon}$, then by partial summation, from the above result and the estimate $\tau(\chi)\ll q^{\frac{1}{2}}$, we obtain

$$S_n(\alpha)=\frac{\mu(q)}{\varphi(q)}T_n(\eta)+O(UP^{1/2}L^{D-C_1}),$$

uniformly for $\eta\in\xi_q$, $q\le P$ and $(a,q)=1$.

Then we write

$$\int_{I_{a,q}}S_n(\alpha)^2e(-2n\alpha)d\alpha=$$

$$\frac{\mu(q)^2}{\varphi(q)^2}e\left(-\frac{2na}{q}\right)\int_{\xi_q}T_n(\eta)^2e(-2n\eta)d\eta+O(UPL^{-2C_1+3D})=$$

$$\frac{\mu(q)^2}{\varphi(q)^2}e\left(-\frac{2na}{q}\right)\int_0^1T_n(\eta)^2e(-2n\eta)d\eta+O(UPL^{-2C_1+3D})+O\Big(\frac{UL^{-D}}{\varphi(q)^2}\Big).$$

Therefore, we have

$$\int_{\mathfrak{M}}S_n(\alpha)^2e(-2n\alpha)d\alpha=2U\mathfrak{S}(2n,P)+O(T_1)+O(T_2),$$

where

$$\mathfrak{S}(2n,P)=\sum_{q\le P}\sum_{a=1}^{q}{}^{*}\frac{\mu(q)^2}{\varphi(q)^2}e\left(-\frac{2na}{q}\right),$$

$$T_1=UP^3L^{-2C_1+3D},\quad T_2=UL^{-D+1}.$$

Then, by classical arguments [11, ch.3], we get

$$\sum_{N\le 2n\le 2N}|\mathfrak{S}(2n,P)-\mathfrak{S}(2n)|^2\ll NP^{-1}L^2,$$

where

$$\mathfrak{S}(2n)=\sum_{q=1}^{\infty}\sum_{a=1}^{q}{}^{*}\frac{\mu(q)^2}{\varphi(q)^2}e\left(-\frac{2na}{q}\right).$$

Finally we obtain

$$\sum_{\mathfrak{M}} = \sum_{N\le 2n\le 2N} \left| \int_{\mathfrak{M}} S_n(\alpha)^2 e(-2n\alpha)d\alpha - 2U\mathfrak{S}(2n) \right|^2 \ll$$

$$U^2 \sum_{N\le 2n\le 2N} |\mathfrak{S}(2n,P) - \mathfrak{S}(2n)|^2 +$$

$$+ \sum_{N\le 2n\le 2N} \left| \int_{\mathfrak{M}} S_n(\alpha)^2 e(-2n\alpha)d\alpha - 2U\mathfrak{S}(2n,P) \right|^2$$

$$\ll NU^2P^{-1}L^2 + NU^2P^6L^{-4C_1+6D} + NU^2L^{-2D+2} \ll NU^2L^{-A},$$

provided

$$C > A+2, \quad 2D > A+2, \quad 4C_1 > A + 6(C+D).$$

5. Minor arcs

In this section we prove that (3.2) holds.

Let us consider integers $n_1, n_2, ..., n_T$ such that

$$N/2 \le n_1 < n_2 < ... < n_T \le N \text{ and } \left[\frac{N}{2}, N\right] \subset \bigcup_{i=1}^{T} I_i,$$

with $I_i = [n_i - H, n_i + H]$. For every $n \in I_i$, we set

$$S_n(\alpha) - S_{n_i}(\alpha) = S^+(\alpha) + S^-(\alpha),$$

where

$$S^+(\alpha) = \begin{cases} S(\alpha; n_i - U, n_i - n), & \text{if } n < n_i, \\ 0, & \text{if } n = n_i, \\ S(\alpha; n + U, n - n_i), & \text{if } n_i < n, \end{cases}$$

$$S^-(\alpha) = \begin{cases} -S(\alpha; n_i + U, n_i - n), & \text{if } n < n_i, \\ 0, & \text{if } n = n_i, \\ -S(\alpha; n - U, n - n_i), & \text{if } n_i < n. \end{cases}$$

Then we write

$$\left| \int_{\mathfrak{m}} S_n(\alpha)^2 e(-2n\alpha)d\alpha \right|^2 \ll$$

$$\left|\int_{\mathfrak{m}} S_{n_i}(\alpha)^2 e(-2n\alpha)d\alpha\right|^2 + \left|\int_{\mathfrak{m}} S^+(\alpha)^2 e(-2n\alpha)d\alpha\right|^2 +$$

$$\left|\int_{\mathfrak{m}} S^-(\alpha)^2 e(-2n\alpha)d\alpha\right|^2 + \left|\int_{\mathfrak{m}} S^+(\alpha)S^-(\alpha) e(-2n\alpha)d\alpha\right|^2 +$$

$$\left|\int_{\mathfrak{m}} S_{n_i}(\alpha)S^+(\alpha) e(-2n\alpha)d\alpha\right|^2 + \left|\int_{\mathfrak{m}} S_{n_i}(\alpha)S^-(\alpha) e(-2n\alpha)d\alpha\right|^2.$$

Using the Cauchy-Schwarz inequality and Parseval's identity, we obtain

$$\left|\int_{\mathfrak{m}} S^+(\alpha)^2 e(-2n\alpha)d\alpha\right|^2 \ll H^2L^4,$$

$$\left|\int_{\mathfrak{m}} S^-(\alpha)^2 e(-2n\alpha)d\alpha\right|^2 \ll H^2L^4,$$

$$\left|\int_{\mathfrak{m}} S^+(\alpha)S^-(\alpha) e(-2n\alpha)d\alpha\right|^2 \ll H^2L^4,$$

$$\left|\int_{\mathfrak{m}} S_{n_i}(\alpha)S^+(\alpha) e(-2n\alpha)d\alpha\right|^2 \ll UHL^4,$$

$$\left|\int_{\mathfrak{m}} S_{n_i}(\alpha)S^-(\alpha) e(-2n\alpha)d\alpha\right|^2 \ll UHL^4.$$

Then we may write

$$\sum_{\mathfrak{m}} \ll \sum_{i=1}^{T}\sum_{n\in I_i}\left|\int_{\mathfrak{m}} S_n(\alpha)^2 e(-2n\alpha)d\alpha\right|^2 \ll \tag{5.1}$$
$$\sum_{i=1}^{T}\sum_{n\in I_i}\left|\int_{\mathfrak{m}} S_{n_i}(\alpha)^2 e(-2n\alpha)d\alpha\right|^2 + NUHL^4.$$

Since $N^{5/8+\varepsilon} \le U \le N$, from Theorems 2 and 3 of [14], we obtain that, for any $B' > 0$,

$$S_{n_i}(\alpha) = S(\alpha; n_i + U, 2U) \ll UL^{-B'}, \quad \forall\alpha \in \mathfrak{m}.$$

Then, by Bessel's inequality and Parseval's identity, we conclude that

$$\sum_{n\in I_i}\left|\int_{\mathfrak{m}} S_{n_i}(\alpha)^2 e(-2n\alpha)d\alpha\right|^2 \ll \int_{\mathfrak{m}} |S_{n_i}(\alpha)|^4 d\alpha \ll U^3L^{-2(B'-1)}. \tag{5.2}$$

Hence (3.2) follows from (5.1) and (5.2) if $A + 4 < B < 2B' - A - 2$ and Theorem 1 is completely proved.

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Giovanni COPPOLA
D.I.I.M.A., Facoltà di Ingegneria
Università di Salerno
Via Ponte Don Melillo,
84084 Fisciano (Sa), Italy.

Maurizio B.S. LAPORTA
Dipartimento di Matematica e Appl. "R. Caccioppoli"
Complesso Universitario di Monte S. Angelo
Via Cinthia,
80126 Napoli, Italy.