

# ON THE SYMMETRY OF DIVISOR SUMS FUNCTIONS IN ALMOST ALL SHORT INTERVALS

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## Abstract

We study the symmetry of divisor sums functions  $\sigma_{-s}(n) \stackrel{def}{=} \sum_{d|n} d^{-s}$  (for  $\sigma = \operatorname{Re}(s) > 0$ ) in almost all short intervals; by elementary methods (based on the Large Sieve) we give an exact asymptotic estimate for the mean-square (over  $N < x \leq 2N$ ) of their “symmetry sum”  $\sum_{|n-x| \leq h} \operatorname{sgn}(n-x) \sigma_{-s}(n)$  (here  $\operatorname{sgn}(0) = 0$  and  $\operatorname{sgn}(t) \stackrel{def}{=} t/|t|$ , for  $t \neq 0$ ).

## 1. Introduction and statement of the results.

In this paper we study the “symmetry” in “almost all short intervals” of the function

$$\sigma_{-s}(n) \stackrel{def}{=} \sum_{d|n} d^{-s},$$

where  $s \in \mathbf{C}$  has real part  $\sigma > 0$ .

As usual, we say that something holds for “almost all” short intervals  $[x-h, x+h]$ , as  $N < x \leq 2N$ , if it’s true  $\forall x \in ]N, 2N]$ , with at most possibly  $o(N)$  exceptions; and  $[x-h, x+h]$  is “short” whenever  $h = h(N)$  is increasing,  $h \rightarrow \infty$  and  $h = o(N)$  as  $N \rightarrow \infty$ .

In order to study the symmetry of distribution of  $\sigma_{-s}(n)$  around  $x$ , as  $n \in [x-h, x+h]$ , we define,  $\forall s \in \mathbf{C}$  with  $\sigma > 0$ , the “symmetry sum”

$$S^\pm(x) \stackrel{def}{=} \sum_{|n-x| \leq h} \operatorname{sgn}(n-x) \sum_{d|n} d^{-s}$$

and we estimate its mean-square over the segment  $N < x \leq 2N$ , i.e. its “symmetry integral”

$$I_s(N, h) \stackrel{def}{=} \sum_{x \sim N} |S^\pm(x)|^2.$$

Here and hereafter  $x \sim N$  means  $N < x \leq 2N$ .

The problem of the estimation of this symmetry integral has its origin in a paper by Kaczorowski and Perelli [KP(2)], where they give a conditional result for the estimate of the Selberg integral, i.e.

$$J(N, h) \stackrel{def}{=} \int_N^{2N} \left| \sum_{x < n \leq x+h} \Lambda(n) - h \right|^2 dx.$$

(Here  $\Lambda(n)$  is the von-Mangoldt function:  $\Lambda(p^\alpha) = \log p$ , otherwise  $\Lambda(n) = 0$ .)

This integral checks the deviations, on average, of the number of primes in the short interval  $[x, x + h]$  from its expected number; in fact, we can call PNT( $[x, x + h]$ ) the "Prime Number Theorem" in this short interval, i.e. the estimate  $\sum_{x < n \leq x+h} \Lambda(n) \sim h$ ; actually, writing a.a. $x \in [N, 2N]$  to mean almost all, i.e. all, with at most  $o(N)$  possible exceptions, it can be proved that

$$J(N, h) = o(Nh^2) \Leftrightarrow \text{PNT}([x, x + h]) \text{ a.a. } x \in [N, 2N].$$

The problem of PNT in almost all short intervals is very old and the actual state of the art is that we can prove it whenever  $h = N^{1/6-\varepsilon(N)}$ , where  $\varepsilon(N) \rightarrow 0$  as  $N \rightarrow \infty$ , see [Z1].

In passing, we remark that Zaccagnini has found, also, very important consequences of non-trivial bounds for  $J(N, h)$  on the distribution of the zeros of the Riemann  $\zeta$  function, see [Z2].

The estimate given by Kaczorowski and Perelli, then, enabled them to get  $J(N, h) = o(Nh^2)$  in suitable ranges (hence PNT for a.a. short intervals), conditioned to non-trivial bounds for the symmetry integral for the von Mangoldt function, i.e.

$$I(N, h) \stackrel{def}{=} \sum_{x \sim N} \left| \sum_{|n-x| \leq h} \text{sgn}(n-x) \Lambda(n) \right|^2$$

(actually, their definition of  $I(N, h)$  is slightly different, but can be reduced to this one).

Hence, from non-trivial bounds for  $I(N, h)$  they get non-trivial bounds for  $J(N, h)$  (in [KP2], Theorem 2). They prove this link by a new form of the Riemann-von Mangoldt explicit formula, see [KP1]; actually, they find that the main term of the remainders in this formula (like, also, in the classic explicit formula) contains (a form of) the symmetry sum for  $\Lambda(n)$ .

As the problem of finding non-trivial estimates for the symmetry integral of  $\Lambda$  seems hopeless, due to its apparent intractability, the author started to study other arithmetic functions; like  $d(n)$ , the number of divisors of  $n$ , in [CS1], where by the Large Sieve the author and Salerno give asymptotics for the symmetry integral of  $d(n)$ .

This originated the study of, also,  $\omega(n)$ , the number of prime divisors of  $n$ , see [C1]; or even the study of the almost-all symmetry of a class of arithmetic functions, see [CS2].

Also, the author studied the problem of the symmetry of primes giving estimates for the symmetry integral of averages of von Mangoldt functions (but very far from estimating the symmetry integral for  $\Lambda$ , see [C2]).

We hope to study, in the future, the applications of our present estimates to mean values of the Riemann zeta-function, like the moments of  $\zeta(s)$ .

Here we will give an asymptotic for the symmetry integral of

$$\sigma_{-s}(n) = \sum_{d|n} d^{-s},$$

whenever  $\sigma > 0$  (if  $\sigma < 0$ , we "flip" the divisors, like in Dirichlet hyperbola method).

For  $Q = N^{\frac{1}{2+\sigma}}$ , let (hereafter  $\|\alpha\| \stackrel{def}{=} \min_{n \in \mathbf{Z}} |\alpha - n|$ , the distance from integers)

$$D_s(N, h) = 2|\zeta(1+s)|^2 N \sum_{\ell \leq Q} \frac{\mu(\ell)}{\ell^{2+2\sigma}} \sum_{k \leq \frac{Q}{\ell}} \frac{1}{k^{2\sigma}} \left\| \frac{h}{k} \right\|.$$

Then we have (as usual,  $s = \sigma + it$ ,  $\sigma, t \in \mathbf{R}$ ), abbreviating  $L \stackrel{def}{=} \log N$ , the following

**Theorem 1** *Let  $s \in \mathbf{C}$  with  $\sigma > 0$ . Assume that  $h = N^\theta$ , with  $0 < \theta < 1/2$  and  $\theta < \frac{\sigma}{2+\sigma}$ . Then*

$$I_s(N, h) = D_s(N, h) + R(N, h),$$

where,  $\forall \sigma > 0$ ,  $R(N, h) = R(N, h, s) = o(N)$ ; more precisely

$$\begin{aligned} R(N, h) &\ll_s N \left( hL^2 N^{-\frac{\sigma}{\sigma+2}} + hLN^{-\frac{\sigma}{2\sigma+4}} h^{-\sigma} \right) && \text{if } \sigma < 1/2; \\ R(N, h) &\ll_s N \left( hL^2 N^{-\frac{\sigma}{\sigma+2}} + \sqrt{h}L^{3/2} N^{-\frac{\sigma}{2\sigma+4}} \right) && \text{if } \sigma = 1/2; \\ R(N, h) &\ll_s N \left( hL^2 N^{-\frac{\sigma}{\sigma+2}} + \sqrt{h}LN^{-\frac{\sigma}{2\sigma+4}} \right) && \text{if } \sigma > 1/2. \end{aligned}$$

(Here the implied constant may depend on  $s$ ,  $|s|$ ,  $\sigma$  or  $t$ , even on all of them.)

We can also give a more explicit evaluation of the main term, by our next result, for which we need the following

**Definition.**

$$\eta^{(h)}(s) \stackrel{def}{=} \sum_{n=1}^{\infty} \left\| \frac{h}{n} \right\| n^{-s}.$$

**Remark.** We explicitly remark that this series converges  $\forall \sigma > 0$ , but to values (depending on  $h$ ) that may grow to  $\infty$ , as  $h \rightarrow \infty$ .

We'll give the main properties of  $\eta^{(h)}(s)$  while proving our

**Corollary 1** *In the same hypotheses of Theorem 1, if we suppose furthermore  $\theta < \frac{1}{2(\sigma+2)}$ , we get*

$$I_s(N, h) = 2 \frac{|\zeta(1+s)|^2}{\zeta(2+2\sigma)} N \eta^{(h)}(2\sigma) + R(N, h) + \mathcal{O}_s \left( N \left( \frac{1}{h^{2\sigma}} + \frac{h}{N^{\frac{2\sigma}{2+\sigma}}} \right) \right),$$

with the same bounds of Theorem 1 for  $R(N, h)$ .

**Remark.** We emphasize that all the remainders in the Corollary are  $o(N)$ , as ensured by our hypotheses on  $h$ .

The paper is organized as follows:

- in section 2 we give an asymptotic version of the Large Sieve;
- in section 3 we apply it to Theorem 1 and prove Corollary 1.

## 2. An asymptotic version of the Large Sieve.

**Lemma 1** *Let  $A, B$  and  $N$  be natural numbers,  $M$  be an integer and  $c_{j,d}$  be complex numbers ( $\forall j, d \in \mathbf{N}$ ); assume that  $a_n > 0 \forall n \in \mathbf{N}$  and define*

$$\alpha_{j,d}(x) \stackrel{def}{=} \sum_{n \in \mathbf{N}} a_n \chi_{\mathcal{I}(j,d,n)}(x),$$

where  $\mathcal{I}(j, d, n)$  is an interval whose endpoints depend on these three (integer) variables and  $\chi_{\mathcal{I}(j,d,n)}(x)$  indicates its characteristic function; then

$$\begin{aligned} \sum_{x=M+1}^{M+N} \left| \sum_{d=A}^B \sum_{j \leq d}^* \alpha_{j,d}(x) c_{j,d} e_d(jx) \right|^2 &= \sum_{d=A}^B \sum_{j \leq d}^* |c_{j,d}|^2 \sum_{x=M+1}^{M+N} |\alpha_{j,d}(x)|^2 \\ &+ \mathcal{O} \left( \alpha^2 B^2 \log B \sum_{d=A}^B \sum_{j \leq d} |c_{j,d}|^2 \right), \end{aligned}$$

with  $(\alpha > 0)$

$$\alpha \stackrel{def}{=} \max_{\substack{M < x \leq M+N \\ j, d}} |\alpha_{j,d}(x)| \ll 1.$$

(Here the implied constant depends at most on  $A, B, M, N$ ).

For the proof, see [CS1] (also, compare [B]).

**3. Proof of Theorem 1 and of Corollary 1.**

*Proof of Theorem 1.* Let  $\chi_q(x) \stackrel{def}{=} \sum_{\substack{|n-x| \leq h \\ n \equiv 0 \pmod{q}}} \text{sgn}(n-x)$  and, for  $h = o(\sqrt{N})$ ,

$$\begin{aligned} S_f^\pm(x) &\stackrel{def}{=} \sum_{|n-x| \leq h} \text{sgn}(n-x) \sum_{d|n} d^{-s} = \\ &= \sum_{|n-x| \leq h} \text{sgn}(n-x) \sum_{\substack{d|n \\ d \leq \sqrt{n}}} \left( d^{-s} + \left(\frac{n}{d}\right)^{-s} \right) + \mathcal{O}_s \left( N^{-\sigma/2} \left( \frac{h}{\sqrt{N}} + 1 \right) \right) \\ &= \sum_{d \leq \sqrt{x}} \left( d^{-s} \chi_d(x) + \sum_{\left| m - \frac{x}{d} \right| \leq \frac{h}{d}} m^{-s} \text{sgn} \left( m - \frac{x}{d} \right) \right) + \mathcal{O}_s \left( \frac{(h/\sqrt{N} + 1)^2}{N^{\frac{\sigma}{2}}} \right) \\ &= \sum_{d \leq \sqrt{x}} \left( d^{-s} + \left(\frac{x}{d}\right)^{-s} \right) \chi_d(x) + \mathcal{O}_s \left( N^{-\sigma/2} \right), \end{aligned}$$

say; changing name to the variables:

$$\Sigma(x) \stackrel{def}{=} \sum_{q \leq \sqrt{x}} \left( q^{-s} + \left(\frac{x}{q}\right)^{-s} \right) \chi_q(x).$$

Before to apply the Large-Sieve we need to rearrange  $\chi_q(x)$  exponential sum, using its Fourier coefficients property  $c_{at,bt} = \frac{1}{t} c_{a,b}$  (due to the fact that  $dc_{j,d}$  depends only upon  $j/d$ ; also, the mean-value  $c_{d,d}$  is 0)

$$\chi_q(x) = \sum_{j < q} c_{j,q} e_q(jx) = \sum_{d|q} \sum_{\substack{j < q \\ (j,q)=d}} c_{j,q} e_q(jx) = \sum_{d|q} \frac{d}{q} \sum_{\substack{j \leq d \\ (j,d)=1}} c_{j,d} e_d(jx).$$

Hence

$$\begin{aligned} \Sigma(x) &= \sum_{d \leq \sqrt{x}} \left( \sum_{n \leq \frac{\sqrt{x}}{d}} \frac{(nd)^{-s} + (x/(nd))^{-s}}{n} \right) \sum_{j \leq d}^* c_{j,d} e_d(jx) \\ &= \sum_{d \leq \sqrt{2N}} \alpha_d(x) \sum_{j \leq d}^* c_{j,d} e_d(jx), \quad \text{say, where:} \\ \alpha_d(x) &\stackrel{def}{=} d^{-s} \sum_{n \leq \frac{\sqrt{x}}{d}} \frac{1}{n^{1+s}} + \left(\frac{x}{d}\right)^{-s} \sum_{n \leq \frac{\sqrt{x}}{d}} \frac{1}{n^{1-s}}. \end{aligned}$$

By partial summation

$$\sum_{n \leq \frac{\sqrt{x}}{d}} \frac{1}{n^{1-s}} = \frac{(\sqrt{x}/d)^s}{s} + \mathcal{O} \left( \frac{1}{|s|} + 1 + |s| + \left(\frac{\sqrt{x}}{d}\right)^{\sigma-1} \right);$$

also, since  $\sigma > 0$ ,

$$\sum_{n \leq \frac{\sqrt{x}}{d}} \frac{1}{n^{1+s}} = \zeta(1+s) + \mathcal{O}\left(\frac{1}{\sigma} \left(\frac{d}{\sqrt{x}}\right)^\sigma\right),$$

whence, uniformly for all  $d \leq \sqrt{x}$  and uniformly  $\forall x \in [N, 2N]$ , we have

$$\alpha_d(x) = \frac{\zeta(1+s)}{d^s} + \mathcal{O}\left(N^{-\frac{\sigma}{2}} \left[\frac{1}{\sigma} + \frac{1}{|s|} + 1 + |s|\right]\right).$$

Also, in the same range (and same uniformities) we get the bound

$$\alpha_d(x) \ll_s d^{-\sigma} \quad (\text{recall } \sigma > 0).$$

In the following, the symbol  $\mathcal{O}_s$  or, equivalently,  $\ll_s$ , mean a dependence on  $s$  and/or on related quantities, like  $|s|$ ,  $\sigma$  or  $t$ .

Finally, we compute (in the same uniformity ranges)

$$|\alpha_d(x)|^2 = \frac{|\zeta(1+s)|^2}{d^{2\sigma}} + \mathcal{O}_s(d^{-\sigma} N^{-\frac{\sigma}{2}}).$$

Then, in order to use our Lemma 1, we split the range of the moduli  $d$ :

$$\Sigma(x) = \Sigma_1(x) + \Sigma_2(x),$$

say, where

$$\Sigma_1(x) \stackrel{\text{def}}{=} \sum_{d \leq Q} \alpha_d(x) \sum_{j \leq d}^* c_{j,d} e_d(jx) \quad \text{and}$$

$$\Sigma_2(x) \stackrel{\text{def}}{=} \sum_{Q < d \leq \sqrt{2N}} \alpha_d(x) \sum_{j \leq d}^* c_{j,d} e_d(jx).$$

By Lemma 1

$$\begin{aligned} \sum_{x \sim N} |\Sigma_1(x)|^2 &= \sum_{d \leq Q} \sum_{j \leq d}^* |c_{j,d}|^2 \sum_{x \sim N} |\alpha_d(x)|^2 + \mathcal{O}\left(Q^2 L \sum_{d \leq Q} \sum_{j \leq d} |c_{j,d}|^2\right) \\ &= 2 \sum_{d \leq Q} \sum_{\ell | d} \frac{\mu(\ell)}{\ell^2} \left\| \frac{h\ell}{d} \right\| \sum_{x \sim N} \frac{|\zeta(1+s)|^2}{d^{2\sigma}} + \mathcal{O}_s\left(N^{1-\frac{\sigma}{2}} h \sum_{d \leq Q} d^{-\sigma-1}\right) \\ &\quad + \mathcal{O}\left(Q^2 L \sum_{d \leq Q} \frac{h}{d}\right) \\ &= D_s(N, h) + \mathcal{O}_s(N^{1-\frac{\sigma}{2}} h) + \mathcal{O}(Q^2 h L^2), \end{aligned}$$

whose main term ( $D_s$  stands for "Diagonal depending on  $s$ ") is, say,

$$D_s(N, h) \stackrel{def}{=} 2|\zeta(1+s)|^2 N \sum_{\ell \leq Q} \frac{\mu(\ell)}{\ell^{2+2\sigma}} \sum_{k \leq \frac{Q}{\ell}} \frac{1}{k^{2\sigma}} \left\| \frac{h}{k} \right\|.$$

Here, we remark that the form in which we write the diagonal "may change", due to differences in small remainders.

Again by Lemma 1 we have (let  $\alpha = Q^{-\sigma}$ , this time)

$$\begin{aligned} \sum_{x \sim N} |\Sigma_2(x)|^2 &\ll_s N \sum_{Q < d \leq \sqrt{2N}} \frac{h}{d} \left( \frac{1}{d^{2\sigma}} + d^{-\sigma} N^{-\sigma/2} \right) + Q^{-\sigma} NL \sum_{Q < d \leq \sqrt{2N}} \frac{h}{d} \\ &\ll_s NhQ^{-\sigma} L^2. \end{aligned}$$

Then, we choose  $Q$  optimally, by equating the remainders of non-diagonal terms of  $\sum |\Sigma_1|^2$  with these last, due to  $\sum |\Sigma_2|^2$  :

$$Q^2 h L^2 = NhQ^{-\sigma} L^2, \quad \text{i.e.} \quad Q = N^{\frac{1}{\sigma+2}},$$

whence

$$\sum_{x \sim N} |\Sigma_1(x)|^2 + \sum_{x \sim N} |\Sigma_2(x)|^2 = D_s(N, h) + \mathcal{O}_s(NhL^2 N^{-\frac{\sigma}{\sigma+2}}).$$

In order to apply Cauchy inequality (to the  $x$ -mean of  $\Sigma_1(x)\Sigma_2(x)$ ), we need an upper bound for  $D_s$ .

It is now clear that

$$\begin{aligned} D_s(N, h) &\ll_s Nh^{1-2\sigma}, & \forall \sigma \in ]0, 1/2[; \\ D_s(N, h) &\ll_s NL, & \sigma = 1/2; \\ D_s(N, h) &\ll_s N, & \forall \sigma > 1/2. \end{aligned}$$

By the previous estimates on the mean-squares of  $\Sigma_1(x)$  and  $\Sigma_2(x)$  we then get, by applying Cauchy inequality,

$$\begin{aligned} \left| \sum_{x \sim N} |\Sigma(x)|^2 - D_s(N, h) \right| &\ll_s N \left( hL^2 N^{-\frac{\sigma}{\sigma+2}} + hLN^{-\frac{\sigma}{2\sigma+4}} h^{-\sigma} \right), \quad \forall \sigma \in ]0, \frac{1}{2}[; \\ \left| \sum_{x \sim N} |\Sigma(x)|^2 - D_s(N, h) \right| &\ll_s N \left( hL^2 N^{-\frac{\sigma}{\sigma+2}} + \sqrt{h} L^{3/2} N^{-\frac{\sigma}{2\sigma+4}} \right), \quad \sigma = 1/2; \\ \left| \sum_{x \sim N} |\Sigma(x)|^2 - D_s(N, h) \right| &\ll_s N \left( hL^2 N^{-\frac{\sigma}{\sigma+2}} + \sqrt{h} LN^{-\frac{\sigma}{2\sigma+4}} \right), \quad \forall \sigma > 1/2. \end{aligned}$$

Finally, we get Theorem 1, by Cauchy inequality for our earlier estimate:

$$\sum_{x \sim N} |S^\pm(x)|^2 = \sum_{x \sim N} |\Sigma(x)|^2 + \mathcal{O}\left(\sqrt{Nh}\sqrt{N^{1-\sigma}}\right) = D_s(N, h) + R(N, h),$$

where  $R(N, h)$  satisfies the same three bounds (one for each  $\sigma$ -range) as above.  $\square$

*Proof of Corollary 1.*

We first want to render  $D_s$  independent of  $Q$ ; this is accomplished by our hypotheses on  $h$ , which give (in particular)  $h = o(\sqrt{Q})$ .

In fact, under this assumption we get

$$\sum_{\ell \leq Q} \frac{\mu(\ell)}{\ell^{2+2\sigma}} \sum_{k \leq \frac{Q}{\ell}} \frac{1}{k^{2\sigma}} \left\| \frac{h}{k} \right\| = \sum_{\ell \leq h} \frac{\mu(\ell)}{\ell^{2+2\sigma}} \sum_{k \leq \frac{Q}{\ell}} \frac{1}{k^{2\sigma}} \left\| \frac{h}{k} \right\| + \mathcal{O}_s(h^{-2\sigma})$$

and this last sum is, by the choice  $Q = N^{\frac{1}{2+\sigma}}$ ,

$$\begin{aligned} \sum_{\ell \leq h} \frac{\mu(\ell)}{\ell^{2+2\sigma}} \sum_{k=1}^{\infty} \frac{1}{k^{2\sigma}} \left\| \frac{h}{k} \right\| + \mathcal{O}_s\left(\frac{h}{Q^{2\sigma}}\right) &= \\ &= \left( \frac{1}{\zeta(2+2\sigma)} + \mathcal{O}_s\left(\frac{1}{h^{1+2\sigma}}\right) \right) \eta^{(h)}(2\sigma) + \mathcal{O}_s\left(hN^{-\frac{2\sigma}{2+\sigma}}\right), \end{aligned}$$

say, where  $\forall \sigma > 0$

$$\eta^{(h)}(2\sigma) \stackrel{def}{=} \sum_{n=1}^{\infty} \frac{1}{n^{2\sigma}} \left\| \frac{h}{n} \right\| = \sum_{n \leq 2h} \frac{1}{n^{2\sigma}} \left\| \frac{h}{n} \right\| + \mathcal{O}_s(h^{1-2\sigma}) \ll_s h.$$

Hence, by these last estimates, we get,  $\forall \sigma > 0$ , for  $h = N^\theta$ ,  $0 < \theta < \frac{1}{2(\sigma+2)}$

$$\sum_{\ell \leq Q} \frac{\mu(\ell)}{\ell^{2+2\sigma}} \sum_{k \leq \frac{Q}{\ell}} \frac{1}{k^{2\sigma}} \left\| \frac{h}{k} \right\| = \frac{\eta^{(h)}(2\sigma)}{\zeta(2+2\sigma)} + \mathcal{O}_s\left(h^{-2\sigma} + hN^{-\frac{2\sigma}{2+\sigma}}\right)$$

whence, in the same hypotheses,

$$D_s(N, h) = 2 \frac{|\zeta(1+s)|^2}{\zeta(2+2\sigma)} N \eta^{(h)}(2\sigma) + \mathcal{O}_s\left(N \frac{1}{h^{2\sigma}} + N \frac{h}{N^{\frac{2\sigma}{2+\sigma}}}\right).$$

We explicitly remark that the remainders are  $o(N)$ , as  $N \rightarrow \infty$ , as ensured by one of our assumptions on  $h$ , namely  $\theta < \frac{\sigma}{2+\sigma}$ .

As an application, if  $h$  is odd, we get (see above)

$$\eta^{(h)}(2\sigma) > \frac{1}{2^{1+2\sigma}}$$



whence, as  $N \rightarrow \infty$

$$D_s(N, h) \geq \frac{1}{2^{2\sigma}} \frac{|\zeta(1+s)|^2}{\zeta(2+2\sigma)} N.$$

The previous explicit expression of  $D_s$  proves our Corollary 1.  $\square$

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