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ON THE SYMMETRY OF SQUARE-FREE SUPPORTED ARITHMETICAL FUNCTIONS IN SHORT INTERVALS

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ABSTRACT. We study the links between additive and multiplicative arithmetical functions, say f, and their square-free supported counterparts, i.e. $\mu^2 f$ (here μ^2 is the square-free numbers characteristic function), regarding the (upper bound) estimate of their symmetry around x in almost all short intervals $[x-h, x+h]$.

Key words and phrases: Symmetry, Square-free, Short intervals.

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1. INTRODUCTION AND STATEMENT OF THE RESULTS

In this paper we study the symmetry, in almost all short intervals, of square-free supported arithmetical functions.

In our previous paper [3] we applied elementary methods, i.e. the Large Sieve, in order to study the symmetry of distribution (around x) of the square-free numbers in "almost all" the "short" intervals $[x-h, x+h]$ (as usual, "almost all" means for all $x \in [N, 2N]$, except at most $o(N)$ of them; "short" means that $h = h(N)$ and $h \to \infty$, $h = o(N)$, as $N \to \infty$).

As in [1], [2], [4], and [5] on (respectively) the prime-divisors function, von Mangoldt function, the divisor function and a wide class of arithmetical functions, we study the symmetry of our arithmetical function f.

We define the "symmetry sum" of f as (here $\operatorname{sgn}(t) \overset{def}{=} t/|t|$, $\operatorname{sgn}(0) \overset{def}{=} 0$)

$$S_f^{\pm}(x) \overset{def}{=} \sum_{|n-x| \le h} f(n)\operatorname{sgn}(n-x),$$

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and its mean-square as the "symmetry integral" of f:

$$I_f(N,h)\stackrel{def}{=}\sum_{x\sim N}\left|\sum_{|n-x|\le h} f(n)\mathrm{sgn}(n-x)\right|^2.$$

Here and hereafter $x\sim N$ stands for $N<x\le 2N$.

We will connect (in Theorem 1.1 and Theorem 1.2) $I_f(N,h)$ and $I_{\mu^2 f}(N,h)$, for suitable f; thus relating the symmetry of f to that of f on the square-free numbers (μ^2 being their characteristic function). Thus, we can estimate just one symmetry integral for two arithmetical functions, whenever they agree on the square-free numbers.

As an example, for $d(n)$ the divisor function, [4] estimates $I_d(N,h)$; then (using Theorem 1.5 to check the symmetry of $d(n)$ in arithmetic progressions) in Theorem 1.3 we bound $I_{\mu^2 d}(N,h)=I_{\mu^2 2^\Omega}(N,h)$, and then obtain information on $I_{2^\Omega}(N,h)$ by Theorem 1.1 (here the function $2^{\Omega(n)}$ is completely multiplicative, with $2^{\Omega(p)}=2$).

We denote with $\mathcal{F}$ the set of arithmetical functions $f:\mathbb{N}\to\mathbb{C}$ and with $\mathcal{B}$ the set of $f\in\mathcal{F}$, with $|f|$ bounded (by an absolute constant); $\mathcal{M}$ denotes the multiplicative $f\in\mathcal{F}$ and $\mathcal{A}$ the additive ones.

Also, we can define ($\forall\alpha\in]1,2]$) the set of "symmetric" arithmetical functions f as (where we **assume**: $\forall E>0\ \ \sup_{\mathbf{N}}|f|\ll N^E$):

$$\mathcal{S}_\alpha\stackrel{def}{=}\left\{f\in\mathcal{F}:\sup_{q\le N^{c\varepsilon}} I_f(N,h,k,q)\ll\frac{Nh^\alpha}{k^2N^\varepsilon}\forall k\le N^{c\varepsilon},\text{ for some } c,\varepsilon>0\right\}$$

(the $\ll$-constant is absolute, as well as $c>0$), where we have set

$$I_f(N,h,k,q)\stackrel{def}{=}\sum_{x\sim N}\left|\sum_{\substack{|n-x/k|\le h/k\\ n\equiv 0(q)}} f(n)\mathrm{sgn}\left(n-\frac{x}{k}\right)\right|^2;$$

in the following, as here, we willl abbreviate $n\equiv a(q)$ to mean $n\equiv a(\mathrm{mod}\,q)$.

We start giving a first link between f and $\mu^2 f$ (in the sequel $L\stackrel{def}{=}\log N$):

Theorem 1.1. *Let $N,h\in\mathbb{N}$, where $h=h(N)$, $h/L^2\to\infty$ and $h=o(N)$ as $N\to\infty$. Assume $J\ll\frac{\sqrt{h}}{L}$, $J\to\infty$ as $N\to\infty$. Let $\|f\|_\infty:=\sup_{\mathbf{N}}|f|$.*

*If f is **completely multiplicative** then*

(i) $$I_f(N,h)\ll L^2\max_{D\ll J}\sum_{d\sim D} d^2 I_{\mu^2 f}\left(\frac{N}{d^2},\frac{h}{d^2}\right)+\frac{Nh^2}{J^2}\|f\|_\infty^2$$

and

(ii) $$I_{\mu^2 f}(N,h)\ll L^2\max_{D\ll J}\sum_{d\sim D} d^2 I_f\left(\frac{N}{d^2},\frac{h}{d^2}\right)+\frac{Nh^2}{J^2}\|f\|_\infty^2.$$

*If f is **completely additive** then*

(i) $$I_f(N,h)\ll L^2\max_{D\ll J}\sum_{d\sim D} d^2 I_{\mu^2 f}\left(\frac{N}{d^2},\frac{h}{d^2}\right)+\left(\frac{Nh^2}{J^2}+NJ\sqrt{h}L^2\right)\|f\|_\infty^2$$

and

(ii) $$I_{\mu^2 f}(N,h)\ll L^2\max_{D\ll J}\sum_{d\sim D} d^2 I_f\left(\frac{N}{d^2},\frac{h}{d^2}\right)+\left(\frac{Nh^2}{J^2}+NJL^2\right)\|f\|_\infty^2.$$

We generalize Theorem 1.1 to **additive** and to **multiplicative** functions:

Theorem 1.2. *Let $f \in \mathcal{A} \cup \mathcal{M}$. Let N, h be natural numbers, with $h = N^\theta$ (for $0 < \theta < 1$). Assume that f is supported over the cube-free numbers and that $\forall E > 0$, $\|f\|_\infty \ll N^E$, as $N \to \infty$. Choose $\forall \alpha \in]1,2]$ $\varepsilon = \frac{\theta(\alpha-1)}{3} > 0$. Then*

$$f \in \mathcal{S}_\alpha \Leftrightarrow \mu^2 f \in \mathcal{S}_\alpha.$$

We give a **concrete example**: the function $f(n) = 2^{\Omega(n)}$ (where $\Omega(n)$ is the total number of prime divisors of n); in this case $f \in \mathcal{S}_\alpha$ and $\mu^2 f \in \mathcal{S}_\alpha$ $\forall \alpha > \frac{3}{2}$, as we will prove directly, also to detail the (more delicate) estimates

Theorem 1.3. *Let $N, h \in \mathbb{N}$, $h = h(N) \geq L$ and $h = o\left(\frac{\sqrt{N}}{L}\right)$ as $N \to \infty$. Then*

$$\sum_{x \sim N} \left| \sum_{|n-x| \leq h} 2^{\Omega(n)} \operatorname{sgn}(n-x) \right|^2 \ll N h^{3/2} N^\varepsilon$$

and

$$\sum_{x \sim N} \left| \sum_{|n-x| \leq h} \mu^2(n) 2^{\Omega(n)} \operatorname{sgn}(n-x) \right|^2 \ll N h^{3/2} N^\varepsilon.$$

Remark 1.4. We explicitly remark that these bounds are non-optimal.

This result is obtained directly upon estimating the mean-square of the symmetry sum for the divisor function **over** the **arithmetic progressions**:

Theorem 1.5. *Let $N, h \in \mathbb{N}$, with $h = h(N) \to \infty$ and $h = o\left(\frac{\sqrt{N}}{L}\right)$ as $N \to \infty$. Then, uniformly $\forall q \in \mathbb{N}$,*

$$\sum_{x \sim N} \left| \sum_{\substack{|n-x| \leq h \\ n \equiv 0(q)}} d(n) \operatorname{sgn}(n-x) \right|^2 \ll N h L^3 + N L^2 \log^2 q,$$

where the $\mathcal{O}$-constant does not depend on q.

The paper is organized as follows

- In Section 2 we give the necessary lemmas;
- In Section 3 we prove our theorems.

2. LEMMAS

Lemma 2.1. *Let $f \in \mathcal{F}$ be an arithmetical function, $\|f\|_\infty \stackrel{def}{=} \sup_{\mathbf{N}} |f(n)|$.*
Then, for N, $h = h(N) \in \mathbb{N}$ and $h \to \infty$, $h = o(N)$ as $N \to \infty$:

$$\sum_{x \sim N} \left| \sum_{\sqrt{2h} < d \leq \sqrt{x+h}} a(d) \sum_{\left|m - \frac{x}{d^2}\right| \leq \frac{h}{d^2}} b(m) f(md^2) \operatorname{sgn}\left(m - \frac{x}{d^2}\right) \right|^2 \ll N h L^2 \|f\|_\infty^2,$$

uniformly $\forall a, b \in \mathcal{B}$.

(Actually, for our purposes, $\|f\|_\infty = \max_{N-h \leq n \leq 2N+h} |f(n)|$).

Proof. Let Σ be the LHS. By a dyadic dissection and Cauchy inequality

$$\Sigma \ll L^2 \max_{\sqrt{h}\ll D\ll\sqrt{N}} \sum_{x\sim N} \left| \sum_{d\sim D} a(d) \sum_{\left|m-\frac{x}{d^2}\right|\le\frac{h}{d^2}} b(m) f(md^2) \mathrm{sgn}\left(m-\frac{x}{d^2}\right) \right|^2$$

$$\ll L^2 \max_{\sqrt{h}\ll D\ll\sqrt{N}} D \sum_{x\sim N} \sum_{d\sim D} \left| \sum_{\left|m-\frac{x}{d^2}\right|\le\frac{h}{d^2}} b(m) f(md^2) \mathrm{sgn}\left(m-\frac{x}{d^2}\right) \right|^2$$

$$\ll \|f\|_\infty^2 L^2 \max_{\sqrt{h}\ll D\ll\sqrt{N}} D \sum_{d\sim D} \sum_{\frac{N-h}{d^2}\le m_1,m_2\le\frac{2N+h}{d^2}} \sum_{\substack{N<x\le 2N \\ m_1d^2-h\le x\le m_1d^2+h \\ m_2d^2-h\le x\le m_2d^2+h}} 1.$$

Clearly, the limitations on x imply $m_1-\frac{2h}{d^2}\le m_2\le m_1+\frac{2h}{d^2}$ (here we "reflect" the "sporadicity") and this in turn, due to $D\gg\sqrt{h}\Rightarrow d^2\gg h$, gives ($\forall m_1$ FIXED) $\mathcal{O}(1)$ possible values to m_2. Hence Σ is bounded by

$$\|f\|_\infty^2 hL^2 \max_{\sqrt{h}\ll D\ll\sqrt{N}} D \sum_{d\sim D} \sum_{\frac{N-h}{d^2}\le m_1\le\frac{2N+h}{d^2}} \sum_{|m_2-m_1|\ll 1} 1 \ll NhL^2 \|f\|_\infty^2 .$$

□

Lemma 2.2. *Assume* $f\in\mathcal{F}$ *is completely additive and* $\|f\|_\infty \stackrel{def}{=} \sup_{\mathbf{N}}|f|$. *Let* $N,h\in\mathbb{N}$ *with* $h=h(N)\to\infty$, $h=o(N)$, *as* $N\to\infty$. *Then* $\forall J\le\sqrt{2h}$

$$\sum_{x\sim N} \left| \sum_{d\le\sqrt{2h}} a(d) \sum_{\left|m-\frac{x}{d^2}\right|\le\frac{h}{d^2}} b(m) f(md^2) \mathrm{sgn}\left(m-\frac{x}{d^2}\right) \right|^2$$

$$\ll L^2 \max_{D\ll J} D \left(\|f\|_\infty^2 \sum_{d\sim D} \sum_{x\sim N} \left| \sum_{\left|m-\frac{x}{d^2}\right|\le\frac{h}{d^2}} b(m) \mathrm{sgn}\left(m-\frac{x}{d^2}\right) \right|^2 \right.$$

$$\left. + \sum_{d\sim D} \sum_{x\sim N} \left| \sum_{\left|m-\frac{x}{d^2}\right|\le\frac{h}{d^2}} b(m) f(m) \mathrm{sgn}\left(m-\frac{x}{d^2}\right) \right|^2 \right) + \frac{Nh^2}{J^2}\|f\|_\infty^2 ,$$

uniformly $\forall a,b\in\mathcal{B}$ *(bounded arithmetical functions).*

Proof. Let us call the left mean-square Σ. Then Σ is at most

$$\sum_{x\sim N} L^2 \max_{D\ll J} \left| \sum_{d\sim D} a(d) \sum_{\left|m-\frac{x}{d^2}\right|\le\frac{h}{d^2}} b(m) f(md^2) \mathrm{sgn}\left(m-\frac{x}{d^2}\right) \right|^2 + \frac{Nh^2}{J^2}\|f\|_\infty^2 .$$

Since f is completely additive

$$\Sigma \ll L^2 \max_{D\ll J} D \left(\sum_{x\sim N} \sum_{d\sim D} \left| \sum_{\left|m-\frac{x}{d^2}\right|\le\frac{h}{d^2}} b(m) f(m) \mathrm{sgn}\left(m-\frac{x}{d^2}\right) \right|^2 \right.$$

$$+ \|f\|_\infty^2 \sum_{x\sim N} \sum_{d\sim D} \left| \sum_{\left|m-\frac{x}{d^2}\right| \le \frac{h}{d^2}} b(m)\mathrm{sgn}\left(m - \frac{x}{d^2}\right) \right|^2 \Bigg) + \frac{Nh^2}{J^2} \|f\|_\infty^2 ,$$

by the Cauchy inequality. The lemma is thus proved. □

Lemma 2.3. *Let f be completely multiplicative. Then, if $N, h \in \mathbb{N}$, with $h = h(N) \to \infty$ and $h = o(N)$ (as $N \to \infty$), we have $\forall J \le \sqrt{2h}$*

$$\sum_{x\sim N} \left| \sum_{d \le \sqrt{2h}} a(d) \sum_{\left|m-\frac{x}{d^2}\right| \le \frac{h}{d^2}} b(m) f(md^2)\mathrm{sgn}\left(m - \frac{x}{d^2}\right) \right|^2$$
$$\ll \|f\|_\infty^2 \left(L^2 \max_{D\ll J} D \sum_{d\sim D} \sum_{x\sim N} \left| \sum_{\left|m-\frac{x}{d^2}\right| \le \frac{h}{d^2}} b(m) f(m)\mathrm{sgn}\left(m - \frac{x}{d^2}\right) \right|^2 + \frac{Nh^2}{J^2} \right)$$

uniformly $\forall a, b \in \mathcal{B}$.

Proof. Let us call the left mean-square Σ. Then

$$\Sigma \ll \sum_{x\sim N} L^2 \max_{D\ll J} \left| \sum_{d\sim D} a(d) \sum_{\left|m-\frac{x}{d^2}\right| \le \frac{h}{d^2}} b(m) f(md^2)\mathrm{sgn}\left(m - \frac{x}{d^2}\right) \right|^2 + \frac{Nh^2}{J^2} \|f\|_\infty^2 ,$$

and being f completely multiplicative we get

$$\Sigma \ll \|f\|_\infty^2 L^2 \max_{D\ll J} D \sum_{d\sim D} \sum_{x\sim N} \left| \sum_{\left|m-\frac{x}{d^2}\right| \le \frac{h}{d^2}} b(m) f(m)\mathrm{sgn}\left(m - \frac{x}{d^2}\right) \right|^2 + \frac{Nh^2}{J^2} \|f\|_\infty^2 ,$$

by the Cauchy inequality. The lemma is thus proved. □

Lemma 2.4. *Let N, h, J and D be as in Lemma 2.2, with $D = o(\sqrt{h})$. Then*

$$\sum_{d\sim D} \sum_{x\sim N} \left| \sum_{\left|m-\frac{x}{d^2}\right| \le \frac{h}{d^2}} f(m)\mathrm{sgn}\left(m - \frac{x}{d^2}\right) \right|^2$$
$$\ll \sum_{d\sim D} d^2 \sum_{y\sim \frac{N}{d^2}} \left| \sum_{|m-y| \le h/d^2} f(m)\mathrm{sgn}(m-y) \right|^2 + \left(\frac{h^2}{D} + ND \right) \|f\|_\infty^2 .$$

Proof. Write $x = yd^2 + r$ $(0 \le r < d^2)$ and let Σ be the left mean-square; since we have $\sum_{x\sim N} = \sum_{y\sim \frac{N}{d^2}} + \mathcal{O}\left(d^2\right)$, then

$$\Sigma \ll \sum_{d\sim D} \sum_{0\le r<d^2} \sum_{y\sim \frac{N}{d^2}} \left| \sum_{\left|m-y-\frac{r}{d^2}\right| \le \frac{h}{d^2}} f(m)\mathrm{sgn}\left(m - y - \frac{r}{d^2}\right) \right|^2 + \frac{h^2}{D} \|f\|_\infty^2$$

(thus $\frac{h^2}{D}$ is due to x-range remainders); then correcting $\mathcal{O}(1)$ values of the m-sum gives as a remainder (due to h-range)

$$\mathcal{O}\left(\sum_{d\sim D} d^2 \frac{N}{d^2} \|f\|_\infty^2\right) = \mathcal{O}\left(ND\|f\|_\infty^2\right).$$

Gathering the estimates we then obtain the lemma. □

3. PROOF OF THE THEOREMS

We start by proving Theorem 1.1.

Proof. In both cases (f completely additive or completely multiplicative) we use the hypothesis on f to "separate variables" after having expressed the symmetry of f by that of $\mu^2 f$ (for i), say) and the symmetry of $\mu^2 f$ by that of f (for ii), say). Thus, to prove i) it will suffice to remember that each natural number $n = md^2$, where m and d are natural and $\mu^2(m) = 1$, i.e. m is square-free:

$$\sum_{|n-x|\le h} f(n)\mathrm{sgn}(n-x) = \sum_{d\le\sqrt{x+h}} \sum_{\left|m-\frac{x}{d^2}\right|\le\frac{h}{d^2}} \mu^2(m) f(md^2)\mathrm{sgn}\left(m-\frac{x}{d^2}\right).$$

Instead, to prove ii) we simply use the following formula (see [7]):

$$\mu^2(n) = \sum_{d^2|n} \mu(d) \qquad \forall n\in\mathbb{N}$$

to get

$$\sum_{|n-x|\le h} \mu^2(n) f(n)\mathrm{sgn}(n-x) = \sum_{d\le\sqrt{x+h}} \mu(d) \sum_{\left|m-\frac{x}{d^2}\right|\le\frac{h}{d^2}} f(md^2)\mathrm{sgn}\left(m-\frac{x}{d^2}\right).$$

As for the additional terms in the completely additive case, they come from the estimate of the square-free symmetry sum as in [3].

Putting together Lemmas 2.1, 2.2, 2.3 and 2.4, the theorem is proved. □

We now come to the proof of Theorem 1.2.

Proof. We first prove that $f\in\mathcal{S} \Rightarrow \mu^2 f\in\mathcal{S}$.
As before, we split at D (to be chosen); say (here $[a,b]$ is the l.c.m. of a,b)

$$\Sigma \overset{def}{=} \sum_{d\le D}\mu(d) \sum_{\substack{\left|n-\frac{x}{k}\right|\le\frac{h}{k} \\ n\equiv 0([q,d^2])}} f(n)\mathrm{sgn}(n-x)$$

$$= \sum_{d\le D}\mu(d) \sum_{\substack{t|[q,d^2] \\ g=[q,d^2]/t}} \sum_{\substack{\left|m-\frac{x}{kt^2g}\right|\le\frac{h}{kt^2g} \\ (m,g)=1}} f(mt^2g)\mathrm{sgn}\left(m-\frac{x}{kt^2g}\right)$$

and observe that, since f is supported over the cube-free numbers, Σ is

$$\sum_{d\le D}\mu(d)\sum_{\substack{t|[q,d^2]\\ g=[q,d^2]/t}} f(t^2g)\sum_{j|g}\mu(j)\sum_{\substack{\left|m-\frac{x}{kt^2g}\right|\le\frac{h}{kt^2g}\\ m\equiv 0(j)}} f(m)\mathrm{sgn}\left(m-\frac{x}{kt^2g}\right)$$

$$\ll \|f\|_\infty N^\delta \sum_{d\le D}\frac{1}{d}d\max_{j,t\le qd^2}\left|\sum_{\substack{\left|m-\frac{x}{kt[q,d^2]}\right|\le\frac{h}{kt[q,d^2]}\\ m\equiv 0(j)}} f(m)\mathrm{sgn}\left(m-\frac{x}{kt[q,d^2]}\right)\right|,$$

by (see [7]) the estimate $\forall\delta>0\ d(n)\ll n^\delta$; using the hypothesis $f\in\mathcal{S}_\alpha$ we get, by Cauchy inequality

$$\sum_{x\sim N}|\Sigma|^2\ll\|f\|_\infty^2 N^{2\delta}\sum_{d\le D}\frac{1}{d^2}\sum_{d\le D}d^2\frac{Nh^\alpha}{k^2d^4N^\varepsilon}\ll\frac{Nh^\alpha}{k^2N^\varepsilon}$$

Hence, it remains to prove that the mean-square of, say

$$\Sigma'\overset{def}{=}\sum_{D<d\le\sqrt{x+h}}\mu(d)\sum_{\substack{\left|n-\frac{x}{k}\right|\le\frac{h}{k}\\ n\equiv 0([q,d^2])}} f(n)\mathrm{sgn}(n-x)$$

is

$$\sum_{x\sim N}|\Sigma'|^2\ll\frac{Nh^\alpha}{k^2N^\varepsilon}.$$

By the Cauchy inequality and a "sporadicity" argument as in the proof of Lemma 2.1,

$$\sum_{x\sim N}|\Sigma'|^2\ll\|f\|_\infty^2\sum_{x\sim N}\left(\sum_{D<d\le\sqrt{\frac{h}{k}}}\left(\frac{h}{kd^2}+1\right)\right)^2$$

$$+\|f\|_\infty^2 L^2\max_{\sqrt{\frac{h}{k}}\ll J\ll\sqrt{N}} J\sum_{d\sim J}\sum_{x\sim N}\left(\sum_{\left|m-\frac{x}{k[d^2,q]}\right|\le\frac{h}{k[d^2,q]}}1\right)^2$$

$$\ll N^\delta N\left(\frac{h^2}{k^2D^2}+\frac{h}{k}\right)+N^\delta\max_{\sqrt{\frac{h}{k}}\ll J\ll\sqrt{N}} J\sum_{d\sim J}\sum_{\frac{N-h}{k[d^2,q]}<m\le\frac{2N+h}{k[d^2,q]}} h.$$

Hence

$$\sum_{x\sim N}|\Sigma'|^2\ll\frac{Nh^\alpha}{k^2N^\varepsilon}\left(\frac{N^{\delta+\varepsilon}h^{2-\alpha}}{D^2}+h^{1-\alpha}N^{\delta+\varepsilon}k\right).$$

In order to obtain the above required estimate we need $\varepsilon\le\frac{\theta(\alpha-1)}{3}$ (for the II term in brackets) and, comparing the mean-squares of Σ and of Σ', we come to the choice $D=N^{\frac{4-\alpha}{2(\alpha-1)}\varepsilon}$ (I term). This proves the first implication.

As for the reverse implication $\mu^2 f\in\mathcal{S}\Rightarrow f\in\mathcal{S}$ we do not need the hypothesis on the support of f and we use the same method (but using $n=md^2$ instead of the identity for μ^2). This finally proves Theorem 1.2. □

We now prove Theorem 1.5.

Proof. First of all, let us call $I_q(N,h)$ the mean-square to evaluate.

We will closely follow the proof of Theorem 1 in [4].

In fact, we start from the "flipping" property to write:

$$\sum_{\substack{|n-x|\le h\\ n\equiv 0(q)}} d(n)\mathrm{sgn}(n-x) = \frac{1}{q}\sum_{r\le q}\sum_{|n-x|\le h} e_q(rn)\left(2\sum_{\substack{d|n\\ d\le\sqrt{n}}} 1\right)\mathrm{sgn}(n-x) + \mathcal{O}\left(\frac{h}{\sqrt{N}}+1\right),$$

having used the orthogonality of the additive characters (see [7]). By our hypothesis on h (see [4] for the details)

$$\sum_{\substack{|n-x|\le h\\ n\equiv 0(q)}} d(n)\mathrm{sgn}(n-x) = \frac{2}{q}\sum_{r\le q}\sum_{d\le\sqrt{x}}\sum_{\substack{|n-x|\le h\\ n\equiv 0(d)}} e_q(rn)\mathrm{sgn}(n-x) + \mathcal{O}(1)$$

(here the constant is independent of q, like all the others following).

Next, write $n-x=s$ to get (again by orthogonality)

$$\begin{aligned}\sum_{\substack{|n-x|\le h\\ n\equiv 0(d)}} e_q(rn)\mathrm{sgn}(n-x) &= e_q(rx)\sum_{\substack{|s|\le h\\ s\equiv -x(d)}} e_q(rs)\mathrm{sgn}(s)\\ &= \frac{e_q(rx)}{d}\sum_{j\le d} e_d(jx)\sum_{|s|\le h} e_q(rs)e_d(js)\mathrm{sgn}(s)\\ &= e_q(rx)\sum_{j\le d} c_{j,d}(q,r)e_d(jx),\end{aligned}$$

say, where

$$c_{j,d}(q,r)\overset{def}{=}\frac{2i}{d}\sum_{s\le h}\sin\left(2\pi s\left(\frac{r}{q}+\frac{j}{d}\right)\right).$$

Here (w.r.t. the quoted [4, Theorem 1]) we have the dependence of the Fourier coefficients on q and r; also, while $c_{d,d}=0$ there, here (by the estimate in of [6, Chap. 25])

$$c_{d,d}(q,r) = \frac{2i}{d}\sum_{s\le h}\sin\frac{2\pi sr}{q} \ll \frac{q}{rd}.$$

Hence, this term's contribute to the mean-square $I_q(N,h)$ is:

$$\sum_{x\sim N}\left|\frac{1}{q}\sum_{r\le q} e_q(rx)\sum_{d\le\sqrt{x}} c_{d,d}(q,r)e_d(jx)\right|^2 \ll \sum_{x\sim N}\left(\sum_{r\le q}\frac{1}{r}L\right)^2 \ll NL^2\log^2 q$$

(that is why we have this additional remainder, here!).

Henceforth, we can rely upon the proof of [4, Theorem 1], the only difference being the r,s dependence:

$$(*)\quad \sum_{x\sim N}\left|\frac{1}{q}\sum_{r\le q} e_q(rx)\sum_{d\le\sqrt{x}}\sum_{j<d} c_{j,d}(q,r)e_d(jx)\right|^2 \ll \frac{1}{q}\sum_{r\le q}\sum_{x\sim N}\left|\sum_{d\le\sqrt{x}}\sum_{j<d} c_{j,d}(q,r)e_d(jx)\right|^2$$

(we have used the Cauchy inequality).

We apply, then, exactly the same estimates; while there we get (we are quoting inequalities to ease comparison)

$$\sum_{j<d}|c_{j,d}|^2 \le \sum_{j\le d}|c_{j,d}|^2 \le \frac{2h}{d},$$

here we have (the constant $c>0$ is ininfluent)

$$\begin{aligned}\sum_{j\le d}|c_{j,d}(q,r)|^2 &= c\frac{1}{d^2}\sum_{|s_1|,|s_2|\le h}\mathrm{sgn}(s_1)\mathrm{sgn}(s_2)\sum_{j\le d} e\left((s_1-s_2)\left(\frac{r}{q}+\frac{j}{d}\right)\right)\\ &= \frac{c}{d}\sum_{|s_1|\le h}\mathrm{sgn}(s_1)\sum_{\substack{|s_2|\le h\\ s_2\equiv s_1(d)}}\mathrm{sgn}(s_2)e_q(r(s_1-s_2)),\end{aligned}$$

whence, by (*), we get (see [4, Theorem 1]), ignoring the remainder $\mathcal{O}(NL^2\log^2 q)$:

$$\begin{aligned}I_q(N,h) &\ll \frac{1}{q}\sum_{r\le q}NL^2\sum_{d\le\sqrt{2N}}\frac{1}{d}\sum_{|s_1|\le h}\mathrm{sgn}(s_1)\sum_{\substack{|s_2|\le h\\ s_2\equiv s_1(d)}}\mathrm{sgn}(s_2)e_q(r(s_1-s_2))\\ &= NL^2\sum_{d\le\sqrt{2N}}\frac{1}{d}\sum_{|s_1|\le h}\mathrm{sgn}(s_1)\sum_{\substack{|s_2|\le h\\ s_2\equiv s_1(d)\\ s_2\equiv s_1(q)}}\mathrm{sgn}(s_2)\\ &\ll NL^2\left(\sum_{\substack{d\le\frac{h}{L}\\ [d,q]\le\frac{h}{L}}}\frac{1}{d}h+\sum_{\frac{h}{L}<d\le\sqrt{2N}}\frac{1}{d}\left(\frac{h^2}{d}+h\right)\right).\end{aligned}$$

Thus

$$I_q(N,h)\ll NhL^3+NL^2\log^2 q.$$

□

We now prove Theorem 1.3.

Proof. We first show the second estimate.

First of all, we observe that $\mu^2(n)2^{\Omega(n)}=\mu^2(n)d(n)$, $\forall n\in\mathbb{N}$; here we will apply the flipping property of the divisor function as in [4].

Then, we will try to link our symmetry integral (for $\mu^2 2^{\Omega}$) with that of $d(n)$.

Writing $\mu^2(n)$ as before

$$\sum_{|n-x|\le h}\mu^2(n)d(n)\mathrm{sgn}(n-x)=\sum_{d\le\sqrt{x+h}}\mu(d)\sum_{\substack{|n-x|\le h\\ n\equiv 0(d^2)}}d(n)\mathrm{sgn}(n-x).$$

Splitting the range at $D=D(x)\le\sqrt{x+h}$ (to be chosen later), we treat, say

$$\Sigma_1(x)\overset{def}{=}\sum_{d\le D}\mu(d)\sum_{\substack{|n-x|\le h\\ n\equiv 0(d^2)}}d(n)\mathrm{sgn}(n-x)$$

by the Cauchy inequality and Theorem 1.5 to get

$$\sum_{x\sim N} |\Sigma_1(x)|^2 \ll D \sum_{d\le D} \sum_{x\sim N} \left| \sum_{\substack{|n-x|\le h \\ n\equiv 0(d^2)}} d(n)\mathrm{sgn}(n-x) \right|^2$$
$$\ll ND^2L^3(h+L) \ll ND^2hL^3,$$

by our hypothesis on h. It remains to bound the mean-square of, say

$$\Sigma_2(x) \stackrel{def}{=} \sum_{D<d\le\sqrt{x+h}} \mu(d) \sum_{\substack{|n-x|\le h \\ n\equiv 0(d^2)}} d(n)\mathrm{sgn}(n-x).$$

We split again at $\sqrt{2h}$ (to distinguish non-sporadic and sporadic terms).

Since by the classical estimate $d(n) \ll n^\varepsilon$ (see [7]; here $\varepsilon > 0$ will not be the same at each occurrence) we estimate trivially (the non-sporadic terms)

$$\sum_{D<d\le\sqrt{2h}} \mu(d) \sum_{\substack{|n-x|\le h \\ n\equiv 0(d^2)}} d(n)\mathrm{sgn}(n-x) \ll \sum_{D<d\le\sqrt{2h}} \frac{hN^\varepsilon}{d^2} \ll \frac{Nh^2}{D^2} N^\varepsilon$$

we get, together with (the sporadic terms, treated by Lemma 2.1)

$$\sum_{x\sim N} \left| \sum_{\sqrt{2h}<d\le\sqrt{x+h}} \mu(d) \sum_{\substack{|n-x|\le h \\ n\equiv 0(d^2)}} d(n)\mathrm{sgn}(n-x) \right|^2 \ll NhN^\varepsilon,$$

that

$$\sum_{x\sim N} |\Sigma_2(x)|^2 \ll \left(\frac{Nh^2}{D^2} + Nh \right) N^\varepsilon.$$

Thus, comparing the mean-squares of $\Sigma_1(x)$ and $\Sigma_2(x)$ we make the best choice $D = h^{1/4}$, finally proving the second estimate.

Writing I_{2^Ω} for the symmetry integral of 2^Ω, we apply Theorem 1.1 to this function; then, i) gives us

$$I_{2^\Omega}(N,h) \ll L^2 \max_{D\ll J} \sum_{d\sim D} d^2 \frac{N}{d^2} \frac{h^{3/2}}{d^3} N^\varepsilon + \frac{Nh^2}{J^2} N^\varepsilon \ll Nh^{3/2} N^\varepsilon,$$

by the choice $J = \sqrt{h}$. This gives the first estimate, hence finally proving Theorem 1.3. □

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