

On the Selberg integral of the three-divisor function d_3

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1. Introduction and statement of the results.

Recall that $d_3(n) \stackrel{def}{=} \sum_{abc=n} 1$ is the number of ways to write n as a product of three positive integers. Namely, the function d_3 is generated by the Dirichlet series $\zeta(s)^3$, where ζ is the Riemann zeta function. Given positive integers N and $H = o(N)$ as $N \rightarrow \infty$, the SELBERG INTEGRAL of d_3 is the mean-square

$$J_3(N, H) \stackrel{def}{=} \sum_{N < x \leq 2N} \left| \sum_{x < n \leq x+H} d_3(n) - M_3(x, H) \right|^2,$$

where $M_3(x, H) \stackrel{def}{=} H \operatorname{Res}_{s=1} \zeta(s)^3 x^{s-1}$ is the SHORT INTERVALS MEAN-VALUE, i.e. the *expected value* of the d_3 -short sum.

Improving on results in [CL], we give here a new non-trivial bound for $J_3(N, H)$ by comparing it to the related MODIFIED SELBERG INTEGRAL,

$$\tilde{J}_3(N, H) \stackrel{def}{=} \sum_{N < x \leq 2N} \left| \sum_{0 \leq |n-x| \leq H} \left(1 - \frac{|n-x|}{H}\right) d_3(n) - M_3(x, H) \right|^2,$$

where the same choice for the s.i. mean-value $M_3(x, H)$ is due to elementary reasons (see [CL] introduction).

In what follows, the symbols O and \ll are respectively the usual Landau and Vinogradov notations, while we adopt the “modified Vinogradov notation” defined as

$$A(N, H) \lll B(N, H) \stackrel{def}{\iff} \forall \varepsilon > 0, \quad A(N, H) \ll_{\varepsilon} N^{\varepsilon} B(N, H).$$

As any divisor function, d_3 is bounded asymptotically by every arbitrarily small power of the variable, i.e. it satisfies the definition:

$$f \text{ ESSENTIALLY BOUNDED } \stackrel{def}{\iff} \forall \varepsilon > 0 \quad f(n) \ll_{\varepsilon} n^{\varepsilon},$$

and we shortly write $f \lll 1$. According to the “modified Vinogradov notation”, we use to write $d_3 \lll 1$ when d_3 is restricted to $]N - H, 2N + H]$ as before.

The author [C] has proved the lower bound $NH \log^4 N \ll J_3(N, H)$ for $H \ll N^{1/3}$, while, in an attempt to establish a non trivial upper bound, Laporta and the author [CL] have conjectured the following estimate for the modified Selberg integral.

CONJECTURE CL. *If $H \ll N^{1/3}$, then $\tilde{J}_3(N, H) \lll NH$.*

As a consequence one has the main result of the present paper.

THEOREM. *If Conjecture CL holds, then $J_3(N, H) \lll NH^{6/5}$.*

The proof of the Theorem is given in §3, where Conjecture CL is combined with the following general Proposition on the Selberg integral and the modified one of an essentially bounded arithmetic function f , respectively

$$J_f(N, H) \stackrel{def}{=} \sum_{x \sim N} \left| \sum_{x < n \leq x+H} f(n) - M_f(x, H) \right|^2,$$

$$\tilde{J}_f(N, H) \stackrel{def}{=} \sum_{x \sim N} \left| \sum_{0 \leq |n-x| \leq H} \left(1 - \frac{|n-x|}{H}\right) f(n) - M_f(x, H) \right|^2.$$

Here $x \sim N$ means that $N < x \leq 2N$ and the s.i. mean-value is defined as

$$M_f(x, H) \stackrel{def}{=} H \operatorname{Res}_{s=1} F(s) x^{s-1},$$

whenever f is generated by a meromorphic Dirichlet series F with (at most) a pole in $s = 1$. It is not difficult to see that $M_f(x, H) \sim \sum_{x < n \leq x+H} p_c(\log n)$ with good remainders, where p_c is the so called LOGARITHMIC POLYNOMIAL of f of degree $c = \text{ord}_{s=1} F(s) - 1$ (see [CL]). If $M_f(x, H)$ vanishes identically (that is the case when F is regular at $s = 1$), then we say that f is BALANCED.

PROPOSITION. *If $f : \mathbb{N} \rightarrow \mathbb{C}$ is an essentially bounded and balanced function such that*

$$\tilde{J}_f(N, H) \lll NH^{1+A}, \quad \forall H \in [H_1, H_2] \text{ with } N^\delta \ll H_1 \ll H_2 \ll N^{1/2-\delta},$$

for an absolute constant $A \in [0, 1)$ and for a fixed $\delta \in (0, 1/2)$, then

$$J_f(N, H) \lll NH^{1+\frac{1+3A}{5-A}}, \quad \forall H \leq H_2.$$

Beyond Lemma 1 of [CL] (whose formulæ are quoted in §2), for the proof of the Proposition (see §3) it is applied an enhanced version of Gallagher’s Lemma for the exponential sums (see [Ga], Lemma 1) established by the author in the joint paper [CL1] with Laporta. Unfortunately at the moment Conjecture CL remains unproved after a serious gap occurred in the proof of the “Fundamental Lemma” given in a former version of [CL]. Nevertheless, we think it is worthwhile to make further attempts, first for the strongest conjecture $J_3(N, H) \lll NH$, $\forall H \ll N^{1/3}$ seems to be out of reach by any method, then because the bound of the Theorem has an impressive consequence on the 6-th moment of ζ on the critical line (compare [CL] and version 2 of the present paper on arXiv). Actually, the author has already explored in his paper [C2] a link between the Selberg integral $J_k(N, H)$ of the k -divisor function d_k and the $2k$ -th moment of Riemann zeta function on the critical line, i.e.

$$I_k(T) \stackrel{\text{def}}{=} \int_T^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt$$

(under suitably conditions on N, H, T). Somehow the same link holds also between $\tilde{J}_k(N, H)$ and $I_k(T)$, since in Gallagher’s Lemma the right hand side is a Selberg integral de facto and, as showed in [CL1], it can be replaced by the corresponding modified Selberg integral (see Lemma in §2). This is still true for the Dirichlet polynomials case, namely Theorem 1 of [Ga] can be modified in the same fashion. Thus, by applying such a Dirichlet polynomial version of Gallagher’s Lemma within the method of [C2] instead of Theorem 1 of [Ga] one gets the following outstanding consequence of Conjecture CL.

COROLLARY. *If Conjecture CL holds, then $I_3(T) \lll T$.*

In the literature this result is known as the “weak 6-th moment”, since it gives no asymptotic equality for $I_3(T)$, but just an upper bound with additional sufficiently small powers. The proof of the Corollary will be given in a forthcoming paper.

2. Notation and preliminary formulæ.

The CORRELATION of f with SHIFT h is defined as

$$\mathfrak{C}_f(h) \stackrel{\text{def}}{=} \sum_{n \sim N} f(n) \overline{f(n-h)} = \sum_{n \sim N} f(n) \sum_{\substack{m \sim N \\ n-m=h}} \overline{f(m)} + O\left(\max_{[N-|h|, 2N+|h|]} |f|^2 \sum_{n \in [N-|h|, N] \cup [2N, 2N+|h|]} 1 \right).$$

Recall that from the orthogonality of the exponentials $e(\beta) \stackrel{\text{def}}{=} e^{2\pi i \beta}$ ($\beta \in \mathbb{R}$) one has

$$\sum_{n \sim N} f(n) \sum_{\substack{m \sim N \\ n-m=h}} \overline{f(m)} = \int_0^1 |\hat{f}(\alpha)|^2 e(-h\alpha) d\alpha,$$

where

$$\hat{f}(\alpha) \stackrel{\text{def}}{=} \sum_n f(n) e(n\alpha) = \sum_{n \sim N} f(n) e(n\alpha)$$

is truncated in the range $]N, 2N]$ in order to avoid convergence problems. If $f \lll 1$, then (compare [CL])

$$\mathcal{C}_f(h) = \int_0^1 |\widehat{f}(\alpha)|^2 e(-h\alpha) d\alpha + O_\varepsilon(N^\varepsilon |h|), \quad \forall h \neq 0.$$

The CORRELATION of an uniformly bounded *weight* $w : \mathbb{R} \rightarrow \mathbb{C}$, vanishing outside $[-H, H]$, is conveniently defined as

$$\mathcal{C}_w(h) \stackrel{def}{=} \sum_a w(a) \sum_{\substack{b \\ a-b=h}} \overline{w(b)}.$$

In particular, the main weight involved here is the characteristic function of the integers in $[1, H]$, say $u(n) \stackrel{def}{=} \mathbf{1}_{[1, H]}(n)$, whose correlation is

$$\mathcal{C}_u(h) \stackrel{def}{=} \sum_a u(a) \sum_{\substack{b \\ a-b=h}} u(b) = \sum_{a \leq H} \sum_{\substack{b \leq H \\ a-b=h}} 1 = \max(H - |h|, 0).$$

If f is essentially bounded and balanced, then Lemma 1 of [CL] provides the formulæ

$$J_f(N, H) = \sum_h \mathcal{C}_u(h) \mathcal{C}_f(h) + O_\varepsilon(N^\varepsilon H^3), \quad \widetilde{J}_f(N, H) = \sum_h \mathcal{C}_{\mathcal{C}_u/H}(h) \mathcal{C}_f(h) + O_\varepsilon(N^\varepsilon H^3),$$

where we find the *Cesaro weight* $\frac{\mathcal{C}_u(h)}{H} = \max\left(1 - \frac{|h|}{H}, 0\right)$, that is the “normalized” correlation of u . In what follows such formulæ will be applied in the integral form

$$\widetilde{J}_f(N, H) = \int_{-1/2}^{1/2} |\widehat{f}(\alpha)|^2 \frac{|\widehat{u}(\alpha)|^4}{H^2} d\alpha + O_\varepsilon(N^\varepsilon H^3), \quad J_f(N, H) = \int_{-1/2}^{1/2} |\widehat{f}(\alpha)|^2 |\widehat{u}(\alpha)|^2 d\alpha + O_\varepsilon(N^\varepsilon H^3),$$

obtained by using the exponential sum $\widehat{u}(\alpha) \stackrel{def}{=} \sum_{h \leq H} e(h\alpha)$. Note that for every $0 < \varepsilon < 1$ one has

$$(*) \quad |\widehat{u}(\alpha)| > [\varepsilon H] \implies |\alpha| < \frac{1}{2[\varepsilon H]}.$$

Indeed, since $(*)$ is trivial for $\alpha = 0$, we may assume that $0 < \alpha < 1/2$, which easily implies $2\alpha < \sin(\pi\alpha)$. Then, $(*)$ follows also in this case for

$$[\varepsilon H] < |\widehat{u}(\alpha)| = \frac{|\sin(\pi H\alpha)|}{\sin(\pi\alpha)} \leq \frac{1}{\sin(\pi\alpha)} < \frac{1}{2\alpha}.$$

Finally, we quote the aforementioned modified version of Gallagher’s Lemma (see [CL1]).

LEMMA. *Let N, h be positive integers such that $h \rightarrow \infty$ and $h = o(N)$ as $N \rightarrow \infty$. If $f : \mathbb{N} \rightarrow \mathbb{C}$ is essentially bounded and balanced, then*

$$h^2 \int_{-\frac{1}{2h}}^{\frac{1}{2h}} |\widehat{f}(\alpha)|^2 d\alpha \lll \widetilde{J}_f(N, h) + h^3.$$

3. Proofs of the Proposition and the Theorem.

PROOF OF THE PROPOSITION. If H were not essentially bounded, the trivial bound $\lll NH^2$ would imply $J_f(N, H) \lll N$ immediately. Thus, let us assume that $H \gg N^\eta$ for some small $\eta > 0$. Taking $\varepsilon = \varepsilon(H)$, $E = E(H)$ to be determined later such that $0 < \varepsilon < E$ and $\varepsilon, E \rightarrow 0$ as $H \rightarrow \infty$, we write

$$J_f(N, H) \lll \int_{-1/2}^{1/2} |\widehat{f}(\alpha)|^2 |\widehat{u}(\alpha)|^2 d\alpha + H^3 \lll$$

$$\lll \varepsilon^2 H^2 \int_{|\widehat{u}(\alpha)| \leq [\varepsilon H]} |\widehat{f}(\alpha)|^2 d\alpha + E^2 H^2 \int_{[\varepsilon H] < |\widehat{u}(\alpha)| \leq EH} |\widehat{f}(\alpha)|^2 d\alpha + \frac{1}{E^2} \int_{|\widehat{u}(\alpha)| > EH} |\widehat{f}(\alpha)|^2 \frac{|\widehat{u}(\alpha)|^4}{H^2} d\alpha + H^3.$$

By applying Parseval's identity together with (*) one has

$$\begin{aligned} J_f(N, H) &\lll NH^2 \varepsilon^2 + H^2 E^2 \int_{|\alpha| \leq \frac{1}{2h}} |\widehat{f}(\alpha)|^2 d\alpha + \frac{1}{E^2} \int_{-1/2}^{1/2} |\widehat{f}(\alpha)|^2 \frac{|\widehat{u}(\alpha)|^4}{H^2} d\alpha + H^3 \lll \\ &\lll NH^2 \varepsilon^2 + H^2 E^2 \int_{|\alpha| \leq \frac{1}{2h}} |\widehat{f}(\alpha)|^2 d\alpha + \frac{1}{E^2} \widetilde{J}_f(N, H) + \frac{1}{E^2} H^3. \end{aligned}$$

Now the hypothesis $\widetilde{J}_f(N, h) \lll Nh^{1+A}$ and the previous Lemma for $h \stackrel{def}{=} [\varepsilon H] \rightarrow \infty$ imply

$$J_f(N, H) \lll NH^2 \left(\varepsilon^2 + \frac{E^2 H^A}{\varepsilon^{1-A} H} + \frac{H^A}{E^2 H} \right).$$

Taking $\varepsilon = H^{-\frac{2(1-A)}{5-A}}$, $E = H^{-\frac{(1-A)^2}{2(5-A)}}$ one gets $\varepsilon^2 = \frac{E^2 H^{A-1}}{\varepsilon^{1-A}} = \frac{H^{A-1}}{E^2}$, $\varepsilon = o(E)$ and $E \rightarrow 0$, as desired. \square

PROOF OF THE THEOREM. Denoting the logarithmic polynomial of d_3 with p_2 , we recall that $M_3(x, H) \sim \sum_{x < n \leq x+H} p_2(\log n)$ where the implicit remainders give a negligible contribution to $J_3(N, H)$. Thus, it is sufficient to apply the Proposition to the balanced and essentially bounded function $f(n) = d_3(n) - p_2(\log n)$, because Conjecture CL allows to take $A = 0$ and $H_2 \ll N^{1/3}$. \square

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