ON THE SYMMETRY OF ARITHMETICAL FUNCTIONS IN ALMOST ALL SHORT INTERVALS, V

by

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Abstract. We study the symmetry in short intervals of arithmetic functions with non-negative exponential sums.

1. Introduction and statement of the results.

We pursue the study of the symmetry in (almost all) short intervals of arithmetical functions f (see [C1]), where this time we give (non-trivial) results for a new class of such (real) f; the key-property they have is a non-negative exponential sum (see Lemma 2), which is something we require, in order to get a kind of "majorant principle"; this allows us to "smooth" our f into a "restricted" divisor function (see the Theorem), for which (see the Corollary) we apply non-trivial results (both from [C-S], on Acta Arithmetica, and [C2]).

We need the following definitions. Here and in the sequel $h \to \infty$ and $h = o(N)$, if $N \to \infty$.

$$\mathcal{C}_{f_1,f_2}(a) \stackrel{def}{=} \sum_{|a|<n\le N-|a|} f_1(n)f_2(n-a) \quad \forall a \in [-2h,2h]\cap\mathbb{Z}$$

(are the "MIXED" CORRELATIONS of arithmetic, real, $f_1, f_2 : \mathbb{N} \to \mathbb{R}$) SAY $\mathrm{sgn}(r) \stackrel{def}{=} \frac{|r|}{r}$ $\forall r \in \mathbb{R}^*$, $\mathrm{sgn}(0) \stackrel{def}{=} 0$:

$$I_{f_1,f_2}(N,h) \stackrel{def}{=} \int_h^N \Big(\sum_{|n-x|\le h} f_1(n)\mathrm{sgn}(n-x)\Big)\Big(\sum_{|m-x|\le h} f_2(m)\mathrm{sgn}(m-x)\Big) dx$$

(are the "MIXED" SYMMETRY INTEGRAL of the arithmetical, real, functions f_1 & f_2)

$$I_f(N,h) \stackrel{def}{=} \int_h^N \Big| \sum_{|n-x|\le h} f(n)\mathrm{sgn}(n-x)\Big|^2 dx$$

(is the SYMMETRY INTEGRAL of the arithmetical real function $f : \mathbb{N} \to \mathbb{R}$)

REMARK Here I_{f_1,f_2} & I_f (like, also, mixed correlations) depend only on f, f_1, f_2 values in $[1, N+h-1]\cap\mathbb{N}$; for all of these quantities it is essential to assume $h = o(N)$ (if $N \to \infty$), to avoid trivialities.

Write the DIVISOR FUNCTION $d(n) \stackrel{def}{=} \sum_{q|n} 1$ AND $d_Q(n) \stackrel{def}{=} \sum_{q|n, q\le Q} 1$ the "RESTRICTED" DIVISOR FUNCTION (correspondingly I_{f,d_Q} and I_{d_Q}). Also, $*$ is Dirichlet product (see the following) and μ is Möbius function [T]; we'll indicate "supp" for the support of our functions and abbreviate

$$A \lll B \stackrel{def}{\iff} \forall \varepsilon > 0 \;\; A \ll_\varepsilon N^\varepsilon B$$

whence, for example, $d(n) \lll 1$ (i.e., SAY, the divisor function is "ESSENTIALLY BOUNDED").

We define (see Lemma 1) $\quad W(a) \stackrel{def}{=} \max(2h - 3|a|, |a| - 2h) \;\; \forall a \in \mathbb{Z} \cap [-2h, 2h]$, $\;\mathrm{supp}\,(W) \subset [-2h, 2h]$.

Then, we come to our main result, which we refer to as a "MAJORANT PRINCIPLE" for I_f:

THEOREM. *Let $h \to \infty$ and $h = o(N)$ when $N \to \infty$. Assume that the arithmetical function $f : \mathbb{N} \to \mathbb{R}$ (which might, and actually will, depend on N and h) has $\mathrm{supp}\,(f*\mu) \subset [1, Q]$, together with the property*

$$\forall \varepsilon > 0 \quad f(n) \ll_\varepsilon N^\varepsilon \;\; \forall n \ll N \;\; (\text{UNIFORMLY IN } 0 < n \ll N)$$

and indicate with $f(0) > 0$ a constant which may depend on N, h (I.E. $f(0) = f_{N,h}(0) > 0$). *Then*

$$S_f(\alpha) \stackrel{def}{=} \Re \sum_{0\le n\le N} f(n)e(n\alpha) \ge 0 \;\; \forall \alpha \in [0,1] \;\;\Rightarrow\;\; I_f(N,h) \lll I_{f,d_Q}(N,h) + h^3 + f(0)h^2 + Qhf(0) + Qh^2.$$

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An immediate consequence is:

<u>COROLLARY</u>. *In the Theorem hypotheses*, ASSUMING ALSO $h^2 \ll Q$, $\theta \buildrel{def}\over{=} \frac{\log h}{\log N} < \frac{1}{2}$,

$$S_f(\alpha) \geq 0 \ \ \forall \alpha \in [0,1] \Rightarrow I_f(N,h) \lll Nh\sqrt{h} + Qhf(0) + Qh^2.$$

In fact, applying Cauchy-Schwarz inequality $I_{f,d_Q}(N,h) \leq \sqrt{I_f(N,h)}\sqrt{I_{d_Q}(N,h)}$ and

$$f(n) \lll 1 \ \ \Rightarrow \ \ I_f(N,h) \lll Nh^2 \quad \text{is the trivial estimate, while}$$

$$\theta < \frac{1}{2} \ \ \Rightarrow \ \ I_{d_Q}(N,h) \lll Nh, \quad \text{SEE [C-S] \& COMPARE [C2].}$$

All of this gives $I_{f,d_Q}(N,h) \lll Nh^{3/2}$, which is bigger than $\lll h^3$ (due to $\theta < \frac{1}{2} < \frac{2}{3}$, here).

<u>REMARK</u> The feature, both in the Theorem and in the Corollary,

$$S_f(\alpha) \geq 0 \quad \forall \alpha \in [0,1]$$

is too strong, as the requirement might be milder (in order to apply Lemma 2, see following section):

$$S_f\left(\frac{j}{q}\right) \geq 0, \ \ \forall j \in \mathbb{Z}_q^* \ \text{AND} \ \forall q \leq Q$$

(being $\mathbb{Z}_q^*$ the reduced residue classes modulo q, here) suffices for our Theorem (and our Corollary).

We have used and will use the notation, for DIRICHLET PRODUCT

$$(f_1 * f_2)(n) \buildrel{def}\over{=} \sum_{d|n} f_1(d) f_2(n/d) = \sum_{d|n} f_1(n/d) f_2(d) \quad \forall n \in \mathbb{N};$$

then, MÖBIUS INVERSION FORMULA reads $f = g * \mathbf{1} \Leftrightarrow g = f * \mu$, see [T] (whence $f \lll 1 \Leftrightarrow g \lll 1$).

In the sequel $(j,q) = 1$ indicates, as usual, that j, q are coprime (no common prime divisors) and we'll write $j (\text{mod } q)$ for the residue classes modulo q (doesn't matter if $j \leq q$ or $0 \leq |j| \leq q/2$, here).
Also, we' ll follow the standard notation $e(n\alpha) \buildrel{def}\over{=} e^{2\pi i n\alpha}$ $(\forall n \in \mathbb{N} \ \forall \alpha \in \mathbb{R})$ for additive characters.

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We'll start with three Lemmas which (resp.ly) manage correlations, link mixed symmetry integrals with mixed correlations and, finally, give the essence of our, say, "MAJORANT PRINCIPLE".

The paper is organized as follows:
— in section 2 we state and prove our Lemmas;
— in section 3 we prove our Theorem;
— in section 4 we give an example.

2. Lemmas.

<u>LEMMA 0</u>. *Let* $h\to\infty$ *and* $h=o(N)$ *when* $N\to\infty$. *Assume* $a\in\mathbb{Z}$ *with* $0<|a|\le 2h$ *and*

$$f_1:\mathbb{N}\to\mathbb{R},\ f_2:\mathbb{N}\to\mathbb{R}\ \text{ARE SUCH THAT}\ \forall\varepsilon>0\ f_1(n),f_2(n)\ll_\varepsilon N^\varepsilon\ \text{UNIFORMLY}\ \forall n\ll N.$$

Then $\forall\varepsilon>0$

$$\sum_{\substack{2h<n<N-h\\ 2h<n-a<N-h}} f_1(n)f_2(n-a)=\mathcal{C}_{f_1,f_2}(a)+\mathcal{O}_\varepsilon\left(N^\varepsilon h\right)\quad\text{AND}\quad \mathcal{C}_{f_1,f_2}(-a)=\mathcal{C}_{f_2,f_1}(a)+\mathcal{O}_\varepsilon\left(N^\varepsilon h\right).$$

<u>PROOF</u>. Assume $a>0$: LHS becomes (recall $|a|\ll h$ and $f_1,f_2\ll_\varepsilon N^\varepsilon$, here)

$$\sum_{2h+a<n<N-h} f_1(n)f_2(n-a)=\sum_{a<n\le N-a} f_1(n)f_2(n-a)+\mathcal{O}_\varepsilon\left(N^\varepsilon h\right)=\mathcal{C}_{f_1,f_2}(a)+\mathcal{O}_\varepsilon\left(N^\varepsilon h\right);$$

instead, in the range $a<0$, $|a|\ll h$ (actually, $-2h\le a<0$),

$$\sum_{2h<n<N-h+a} f_1(n)f_2(n-a)=\sum_{-a<n\le N+a} f_1(n)f_2(n-a)+\mathcal{O}_\varepsilon\left(N^\varepsilon h\right)=\mathcal{C}_{f_1,f_2}(a)+\mathcal{O}_\varepsilon\left(N^\varepsilon h\right).$$

FINALLY, $$\mathcal{C}_{f_1,f_2}(-a)=\sum_{|a|+a<m\le N-|a|+a} f_1(m-a)f_2(m)=\mathcal{C}_{f_2,f_1}(a)+\mathcal{O}_\varepsilon\left(N^\varepsilon h\right).\ \square$$

<u>LEMMA 1</u>. *Let* $h\to\infty$ *and* $h=o(N)$ *when* $N\to\infty$. *Assume that*

$$f_1:\mathbb{N}\to\mathbb{R}\ \text{AND}\ f_2:\mathbb{N}\to\mathbb{R}\ \text{SATISFY}\ \forall\varepsilon>0\ f_1(n),f_2(n)\ll_\varepsilon N^\varepsilon,\ \text{UNIFORMLY}\ \forall n\ll N.$$

Then $\forall\varepsilon>0$ $$I_{f_1,f_2}(N,h)=\sum_a W(a)\mathcal{C}_{f_1,f_2}(a)+\mathcal{O}_\varepsilon\left(N^\varepsilon h^3\right).$$

<u>PROOF</u>. From the definition, exchanging sums and integral, LHS is, SAY,

$$\sum_{\substack{n,m\le N+h-1\\ 0\le|n-m|\le 2h}}\sum f_1(n)f_2(m)\int_{\substack{h<x<N\\ |x-n|\le h,|x-m|\le h}}\mathrm{sgn}(x-n)\mathrm{sgn}(x-m)dx=\sum_{\substack{n,m\le N+h-1\\ 0\le|n-m|\le 2h}}\sum f_1(n)f_2(m)\mathcal{I}_{N,h}(m,n),$$

since $x=h$ and $x=N$ have no importance (0−measure) in the integral and $|x-n|\le h$, $|x-m|\le h$ give $|n-m|\le 2h$ (from triangle inequality); here the condition $h<x<N$ can be dispensed with into

$$\sum_{\substack{2h<n,m<N-h\\ 0\le|n-m|\le 2h}}\sum f_1(n)f_2(m)\mathcal{I}_{N,h}(m,n)=\sum_{\substack{2h<n,m<N-h\\ 0\le|n-m|\le 2h}}\sum f_1(n)f_2(m)W(|n-m|),\qquad\text{where}$$

$$\int_{\substack{|x-n|\le h\\ |x-m|\le h}}\mathrm{sgn}(x-n)\mathrm{sgn}(x-m)dx=\int_{\max(n-h,m-h)}^{\min(n+h,m+h)}\mathrm{sgn}(x-n)\mathrm{sgn}(x-m)dx=W(|n-m|)$$

(SAY, IN ACCORDANCE WITH W DEFINITION, see the above) and, IN FACT, THIS IS THE MAIN TERM

$$\sum_a\sum_{\substack{2h<n,m<N-h\\ n-m=a}}\sum f_1(n)f_2(m)W(a)=\sum_a W(a)\sum_{\substack{2h<n<N-h\\ 2h<n-a<N-h}} f_1(n)f_2(n-a)=\sum_a W(a)\mathcal{C}_{f_1,f_2}(a)+\mathcal{O}_\varepsilon\left(N^\varepsilon h^3\right)$$

from Lemma 0, with a good remainder; like also the terms completing LHS, above expanded :

$$\left(\sum_{\substack{n\le 2h,\ m\le N+h-1\\ 0\le|n-m|\le 2h}}\sum f_1(n)f_2(m)+\sum_{\substack{m\le N+h-1,\ N-h\le n\le N+h-1\\ 0\le|n-m|\le 2h}}\sum f_2(m)f_1(n)\right)\mathcal{I}_{N,h}(m,n)\ll_\varepsilon N^\varepsilon h^3.\ \square$$

We'll write, in the sequel, $\widehat{W}(\beta)\stackrel{def}{=}\sum_a W(a)e(a\beta)\ \forall\beta\in\mathbb{R}$, the "Discrete Fourier Transform" of our W.

<u>LEMMA 2</u>. *Let* $g:\mathbb{N}\to\mathbb{R}$ *be such that* $\forall\varepsilon>0\ g(q)\ll_\varepsilon N^\varepsilon\ \forall q\le Q$ *and* $Q\ll N$ $(Q\to\infty)$ *if* $N\to\infty$. *Then*

$$S_f(\alpha)\ge 0\ \forall\alpha\in[0,1]\Rightarrow\sum_{q\le Q}\frac{g(q)}{q}\sum_{j(\mathrm{mod}\ q)}\widehat{W}\left(\frac{j}{q}\right)S_f\left(-\frac{j}{q}\right)\lll\sum_{q\le Q}\frac{1}{q}\sum_{j(\mathrm{mod}\ q)}\widehat{W}\left(\frac{j}{q}\right)S_f\left(-\frac{j}{q}\right).$$

The PROOF is immediate and simply applies the hypotheses (recall $\widehat{W}\ge 0$, [(1), LEMMA 4, C1]).

3. Proof of the Theorem.

We start giving the following :
IN OUR HYPOTHESES ON h, N, WRITING (in LEMMA 1) $f_1 = f$, $f_2 = g * \mathbf{1}$, WITH $M := \max \mathrm{supp}\,(g)$,

$$(*) \qquad I_{f,g*\mathbf{1}}(N,h) = \sum_a W(a) \sum_{q \le M} g(q) \sum_{\substack{0 \le |n| \le N \\ n \equiv a (q)}} f(n) + \mathcal{O}_\varepsilon \left(N^\varepsilon \big(f(0)h^2 + Mhf(0) + Mh^2 + h^3 \big) \right)$$

FOR $g, f : \mathbb{N} \to \mathbb{R}$ ANY ESSENTIALLY BOUNDED FUNCTIONS. In fact, from Lemma 1 (see also Lemma 0)

$$I_{f,g*\mathbf{1}}(N,h) = \sum_a W(a) \mathcal{C}_{f,g*\mathbf{1}}(a) + \mathcal{O}_\varepsilon \left(N^\varepsilon h^3 \right) = \sum_a W(a) \sum_{q \le M} g(q) \sum_{\substack{|a| < n \le N - |a| \\ n \equiv a(q)}} f(n) + \mathcal{O}_\varepsilon \left(N^\varepsilon h^3 \right),$$

which is, applying $h = o(N)$, here:

$$\sum_a W(a) \sum_{q \le M} g(q) \sum_{\substack{0 \le n \le N \\ n \equiv a(q)}} f(n) + \mathcal{O}_\varepsilon \left(N^\varepsilon h^3 \right) - \left(\sum_{q \le M} g(q) \right) \left(\sum_{a > 0} W(a) f(a) \right) - f(0) \sum_a W(a) \sum_{q | a} g(q),$$

whence $(*)$. THEN, we USE $(*)$ TWICE, ONE TIME FOR $g = f * \mu$ (to treat I_f) AND, SOON AFTER LEMMA 2, WITH $g = \mathbf{1}$ (supported in $[1, Q]$, to get I_{f,d_Q}). (In what follows, we use $\widehat{W}$, see before Lemma 2.)

IN FACT, $\quad g = f * \mu \Rightarrow I_f = I_{f,g*\mathbf{1}}$ AND $\mathrm{supp}\,(g) \subset [1, Q]$ (see our Theorem hypotheses):

$$I_f(N,h) \lll \left| \sum_{q \le Q} \frac{g(q)}{q} \sum_{j (\mathrm{mod}\ q)} \widehat{W}\left(\frac{j}{q}\right) S_f\left(-\frac{j}{q}\right) \right| + h^3 + f(0)h^2 + Qhf(0) + Qh^2$$

$$\lll \sum_{q \le Q} \frac{1}{q} \sum_{j (\mathrm{mod}\ q)} \widehat{W}\left(\frac{j}{q}\right) S_f\left(-\frac{j}{q}\right) + h^3 + f(0)h^2 + Qhf(0) + Qh^2,$$

from $(*)$ above, the orthogonality [D] of additive characters [V] and Lemma 2, with, SAY,

$$S_f(\alpha) := \sum_{0 \le n \le N} f(n) e(n\alpha),$$

but we restrict to its real part (as defined in our theorem), SINCE

$$W \text{ EVEN} \quad \Rightarrow \quad \widehat{W} \text{ EVEN}.$$

FINALLY, APPLY $g = \mathbf{1}$ (WITH $\mathrm{supp}\,(g) \subset [1, Q]$, HERE) INTO $(*)$. □

4. A first non-trivial (though non-optimal) application.

From the remark above, we may restrict α to $\mathcal{F}_Q := \{ j/q : j \le q, (j,q) = 1, q \le Q \}$ (FAREY FRACTIONS)

$$(*) \qquad S_f(\alpha) \ge 0 \ \ \forall \alpha \in \mathcal{F}_Q \quad \Rightarrow \quad I_f(N,h) \lll I_{f,d_Q}(N,h) + Qf(0)h + Qh^2$$

and (see last remainder) we'll assume $\lambda \stackrel{def}{=} \frac{\log Q}{\log N} < 1$ in our future papers applying the majorant principle.

We want to get the condition $S_f \ge 0$ (see $(*)$ above) and we are turning upside-down the usual approach: instead of taking a class of functions f to study, we start from the requirement $S_f \ge 0$ (at least on $\mathcal{F}_Q$, see above) and want to understand for which class of functions $f : \mathbb{N} \to \mathbb{R}$ does it hold.

The idea behind "MAKING S_f POSITIVE" is very easy: we think of the VALUES $f(n)$, $\forall n \ne 0$, as FIXED, CHOOSING THE MEAN-VALUE $f(0)$ in order TO RENDER the whole sum S_f POSITIVE (actually, non-negative).

The problem, now, is a definite one : CHOOSE $f(0)$ IN ORDER TO ENSURE BOTH $S_f \ge 0$ AND THAT $Qf(0)h$ IS A NON-TRIVIAL REMAINDER (namely, $Qf(0)h \ll Nh^2N^{-\delta}$, for some $\delta > 0$, fixed).

Let's start from $S_f \ge 0$ on $\mathfrak{F}_Q$: see that, $\forall n \neq 0$, $f(n) = \sum_{d|n} g(d)$, supp $(g) \subset [1,Q]$ gives (f EVEN)

$$\sum_{0<|n|\le N} f(n)e(n\alpha) = \sum_{d\le Q} g(d) \sum_{0<|m|\le \frac{N}{d}} e(d\alpha m) := \Sigma_1(\alpha) + \Sigma_2(\alpha)$$

and we distinguish (as a kind of "MAJOR ARCS " & "MINOR ARCS", here), say, with $\alpha = \frac{j}{q}$:

$$\Sigma_1(\alpha) \stackrel{def}{=} \sum_{\substack{d\le Q \\ d\equiv 0 (\mathrm{mod}\ q)}} g(d) \sum_{0<|m|\le \frac{N}{d}} 1 \quad \& \quad \Sigma_2(\alpha) \stackrel{def}{=} \sum_{\substack{d\le Q \\ d\not\equiv 0 (\mathrm{mod}\ q)}} g(d) \sum_{0<|m|\le \frac{N}{d}} e(d\alpha m),$$

where we want an UNIFORM estimate over $\alpha \in \mathfrak{F}_Q$.

Fix $\alpha \in \mathfrak{F}_Q$. Let $\|\alpha\| \stackrel{def}{=} \min_{n\in\mathbb{Z}} |\alpha - n|$, $\forall \alpha \in \mathbb{R}$ is the DISTANCE of the real number α FROM THE INTEGERS. Then, $g \ge 0 \Rightarrow \Sigma_1(\alpha) \ge 0$ and (keeping the hypothesis $f * \mu \stackrel{def}{=} g \ge 0$, now on) we don't care about $\Sigma_1(\alpha)$ value : it's a positive contribute to S_f, doesn't matter how much.

As regards the other sum, recall the hypothesis $f \lll 1$ (f is ESSENTIALLY BOUNDED, on non-zero integers) entails (from f, g definition, see above) $f \lll 1 \Rightarrow g \lll 1$, whence (see [D], chap.26) $\sum_{0<|m|\le \frac{N}{d}} e(d\alpha m) \ll \frac{1}{\|\alpha d\|}$

($\alpha \in \mathfrak{F}_Q \Rightarrow \alpha d$ is not an integer, $\forall d$ not a multiple of q) gives

$$\Sigma_2(\alpha) \ll \sum_{\substack{d\le Q \\ d\not\equiv 0 (\mathrm{mod}\ q)}} g(d) \frac{1}{\left\| \frac{jd}{q} \right\|} \lll \sum_{0<|r|\le \frac{q}{2}} \frac{q}{|r|} \sum_{\substack{d\le Q \\ d\equiv r\overline{j} (\mathrm{mod}\ q)}} 1 \lll q \left(\sum_{0<|r|\le \frac{q}{2}} \frac{1}{|r|} \right) \left(\frac{Q}{q} + 1 \right) \lll Q$$

(recall $g \ge 0$, now), since $(j,q) = 1$ implies $jd \equiv r (\mathrm{mod}\ q) \Leftrightarrow d \equiv r\overline{j} (\mathrm{mod}\ q)$, where $\overline{j} (\mathrm{mod}\ q)$ is such that $\overline{j}j \equiv 1 (\mathrm{mod}\ q)$, i.e., $\overline{j}$ is the RECIPROCAL of $j (\mathrm{mod}\ q)$.

See that we do not know the sign of $\Sigma_2(\alpha)$ (while we force $\Sigma_1(\alpha) \ge 0$ with $g \ge 0$), still we may estimate its "maximum amplitude" and then make $S_f \ge 0$ SIMPLY CHOOSING $f(0) = QN^{2\varepsilon}$ (the number of εs is unimportant, now):

$$g \ge 0 \;\Rightarrow\; S_f(\alpha) = QN^{2\varepsilon} + \Sigma_1(\alpha) + \Sigma_2(\alpha) \ge 0 \quad \forall \alpha \in \mathfrak{F}_Q.$$

Now, thanks to $(*)$, the result becomes (assuming above Theorem hypotheses, in particular $f \lll 1$)

$$g \ge 0 \;\Rightarrow\; I_f(N,h) \lll I_{f,d_Q}(N,h) + Q^2h + Qh^2$$

and resembles very much the results of [C1], though weaker (additional hypothesis $g \ge 0$ & weaker error terms).

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