# ON THE SYMMETRY OF ARITHMETICAL FUNCTIONS IN ALMOST ALL SHORT INTERVALS, III

by

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## 1. Introduction and statement of the results.

We pursue the study of arithmetical functions started in [C-S] (now on, I) and continued in [C3] (say, II). Here we will get general results on a class of arithmetical functions $f$ whose Eratosthenes transform, say $g = f * \mu$, satisfies $g(n) \ll n^{\delta}$, $\forall \delta > 0$ (fixed) and $\sum_{d \le D} g(d) \ll D^{\theta}$, with $0 \le \theta < 1$. In this way, we recover the entire class of $f$ studied in II, since $g(n) \ll n^{-\varepsilon}$, for a fixed $\varepsilon > 0$, implies both our present hypotheses on $g$ (with, even, any $\delta \ge 0$ and, say, $\theta = 1 - \varepsilon$).

Also, we find another way of giving non-trivial bounds for the symmetry of, say, square-free numbers; these were studied in [C2], but with (apart from constants) optimal bounds. Here our methods are more general, but for the function $f(n) = \mu^2(n)$ (= characteristic function of square-free numbers) give less precise results. However, they give suitable non-trivial results, of course, also for $k$-free numbers ($\forall k > 1$).

Thus, our present results strictly generalize the ones obtained in II; also, they clarify the reasons why we were able to get asymptotics in a form similar to singular series (see Theorem 2 of II); finally, they support evidence for the hope of getting similar bounds for the (up to now, unreachable) case $\theta = 1$.

First of all, we define, for $N < x \le 2N$ an integer variable (also, $N \in \mathbb{N}$) and $h \in \mathbb{N}$, with $h \to \infty$, $h = o(N)$, as $N \to \infty$, the **symmetry sum** of a real arithmetical function $f$ as:

$$S_f^{\pm}(x) = S_f^{\pm}(x,h) \stackrel{def}{=} \sum_{|n-x| \le h} \mathrm{sgn}(n-x) f(n)$$

and its (discrete-)mean-square, the **symmetry integral** of $f$, say (now on $x \sim N$ means $N < x \le 2N$)

$$I_f(N,h) \stackrel{def}{=} \sum_{x \sim N} \left| \sum_{|n-x| \le h} \mathrm{sgn}(n-x) f(n) \right|^2 .$$

Our aim is to get non-trivial bounds for this quantity, that checks the symmetry of $f$ in "almost all"(i.e., all $x \sim N$, except at most $o(N)$ of them) short intervals $[x-h, x+h]$. This study takes the first steps in the seminal paper of Kaczorowski and Perelli, see the motivation for introducing $I_f$ as given in [C1].

Abbreviate $L \stackrel{def}{=} \log N$ and $n \equiv r(q) \stackrel{def}{\Longleftrightarrow} n \equiv r(\bmod\ q)$ henceforth, writing $[d,q]$ also for l.c.m.$(d,q)$. We remark that, due to our hypotheses on $g, n, x$, we may assume that $g(d)$ vanishes, whenever $d > 2N + h$.

We can give our first result (the weight $W$ is defined in Lemma 2.0, following section):

**Theorem.** *Let* $N, h \in \mathbb{N}$, *with* $h \to \infty$, $h = o(N)$ *as* $N \to \infty$. *Let* $f : \mathbb{N} \to \mathbb{R}$, $f = g * \mathbf{1}$ *be such that*

$$\forall \delta > 0 \ \ g(n) \ll n^{\delta}, \qquad \textit{and} \qquad \exists \theta \in [0,1[: \ \sum_{d \le D} |g(d)| \ll D^{\theta}, \ \ \forall D \ge 1.$$

*Then* $\forall \delta > 0$

$$I_f(N,h) = 2h \sum_{d,q \le 2N+h} \sum g(d)g(q) \sum_{\substack{n \sim N \\ n \equiv 0([d,q])}} 1 + N \sum_{a \ne 0} W(a) \sum_{(d,q)|a} \sum \frac{g(d)g(q)}{[d,q]} + \mathcal{O}_{\theta}\left( \left( N^{\frac{2\theta}{1+\theta}} h + N + h^2 \right) N^{\delta} h \right).$$

*Here the implied constant depends only on* $\theta$ *and can be chosen as* $1/(1-\theta)$.

It follows almost immediately:

**Corollary.** *Let* $N, h \in \mathbb{N}$, *and for, say,* $\eta = \frac{\log h}{\log N}$, *let* $0 < \eta < 1$. *Let* $f$ *be as in the Theorem. Then*

$$I_f(N,h) \ll N h^2 \frac{N^{-\varepsilon}}{1-\theta}$$

*for a suitable* $\varepsilon > 0$ (*depending on* $h$ *and* $\theta$).

In the next section we give the Lemmas, to prove the Theorem in section 3 (the proof of the Corollary is derived easily). Then, in section 4, we make some comments and remarks.

## 2. Lemmas.

We define the (Auto-)Correlation of an arithmetical (real) function $f$ as:

$$\mathcal{C}_f(a) = \mathcal{C}_f(a, N) \stackrel{def}{=} \sum_{n\sim N} f(n)f(n-a), \quad \forall a \in \mathbb{Z}.$$

Our next Lemma gives asymptotics for the symmetry integral of $f$ in terms of its correlations:

**Lemma 2.0.** *Let $N, h \in \mathbb{N}$, with $h \to \infty$, $h = o(N)$ as $N \to \infty$. Let $f$ be a real arithmetic function. Then*

$$I_f(N,h) = \sum_{a\in\mathbb{Z}} W(a)\mathcal{C}_f(a) + \mathcal{O}\left((Nh + h^3)M_f^2\right),$$

*with $M_f = \max_{n\le 3N} |f(n)|$, where the weight $W(a)$ is even and defined on non-negative integers as*

$$W(a) \stackrel{def}{=} \begin{cases} 2h - 3a & \text{if } 0 \le a \le h \\ a - 2h & \text{if } h \le a \le 2h \\ 0 & \text{otherwise} \end{cases},$$

*whence its mean-value (over all the integers $a$) is* $\sum_a W(a) = 0$.

*Proof.* This is a kind of dispersion method for the symmetry integral of $f$:

$$I_f(N,h) = \sum_{x\sim N} \Big| \sum_{|n-x|\le h} \mathrm{sgn}(n-x)f(n) \Big|^2,$$

but with no "expected mean": the main term "vanishes". Nonetheless, the steps involved are the same. Expand the square and exchange the sums to get (since $f$ is real)

$$I_f(N,h) = \sum\sum_{N-h<n_1,\, n_2\le 2N+h} f(n_1)f(n_2) \sum_{\substack{x\sim N \\ |x-n_1|\le h \\ |x-n_2|\le h}} \mathrm{sgn}(x-n_1)\mathrm{sgn}(x-n_2);$$

the diagonal (i.e. $n_1 = n_2$), is, say, $D_f(N,h) \stackrel{def}{=} \sum_{N-h<n\le 2N+h} f^2(n) \sum_{\substack{x\sim N \\ 0<|x-n|\le h}} 1.$ Also,

$$D_f(N,h) = \sum_{N+h<n\le 2N-h} f^2(n) \sum_{0<|x-n|\le h} 1 + \mathcal{O}\Big( \sum_{N-h<n\le N+h} f^2(n)h + \sum_{2N-h<n\le 2N+h} f^2(n)h \Big)$$

$$= 2h \sum_{n\sim N} f^2(n) + \mathcal{O}(h^2 M_f^2) = W(0)\mathcal{C}_f(0) + \mathcal{O}(h^2 M_f^2),$$

from $W$ and $\mathcal{C}_f$ definitions. The remainder, here, is negligible.

We isolate the diagonal ($a = 0$), then compare correlations with opposite "shifts" ($a$ and $-a$): for $a > 0$

$$\mathcal{C}_f(-a) = \sum_{N<n\le 2N} f(n)f(n+a) = \sum_{N+a<m\le 2N+a} f(m-a)f(m)$$

$$= \sum_{N<m\le 2N} f(m)f(m-a) + \mathcal{O}\Big(\Big( \sum_{N<m\le N+a} + \sum_{2N<n\le 2N+a} \Big)|f(m-a)f(m)|\Big) = \mathcal{C}_f(a) + \mathcal{O}(hM_f^2),$$

whence $W(a) \ll h \Rightarrow \sum_{0<a\le 2h} W(a)\mathcal{C}_f(-a) = \sum_{0<a\le 2h} W(a)\mathcal{C}_f(a) + \mathcal{O}(h^3 M_f^2)$ : we'll confine in proving

$$I_f(N,h) - D_f(N,h) = 2\sum_{0<a\le 2h} W(a)\mathcal{C}_f(a) + R_f(N,h), \text{ with, say, } R_f(N,h) = \mathcal{O}((Nh + h^3)M_f^2).$$

We have (resp., from: $f$ is real and the changes of variables $n=n_1$, $a=n_2-n_1$, and then $s=x-n$)

$$I_f(N,h)-D_f(N,h)=2\sum\sum_{N-h<n_1<n_2\le 2N+h} f(n_1)f(n_2)\sum_{\substack{x\sim N\\ |x-n_1|\le h\\ |x-n_2|\le h}} \mathrm{sgn}(x-n_1)\mathrm{sgn}(x-n_2)$$

$$=2\sum_{N-h<n<2N+h} f(n)\sum_{\substack{0<a\le 2h\\ N-h-n<a\le 2N+h-n}} f(n+a)\sum_{\substack{N<x\le 2N\\ |x-n|\le h\\ |x-n-a|\le h}} \mathrm{sgn}(x-n)\mathrm{sgn}(x-n-a)$$

$$=2\sum_{N-h<n<2N+h} f(n)\sum_{\substack{0<a\le 2h\\ a\le 2N+h-n}} f(n+a)\sum_{\substack{N-n<s\le 2N-n\\ |s|\le h\\ |s-a|\le h}} \mathrm{sgn}(s)\mathrm{sgn}(s-a).$$

Now on, we introduce remainders which will take part into our "final" remainder $R_f(N,h)$.
First of all, we simplify the sum over $a$, letting

$$I_f(N,h)-D_f(N,h)=2\sum_{N-h<n\le 2N-h} f(n)\sum_{0<a\le 2h} f(n+a)\sum_{\substack{s>N-n\\ |s|\le h\\ |s-a|\le h}} \mathrm{sgn}(s)\mathrm{sgn}(s-a)+R_1(N,h),$$

where ("tails"-remainder) $R_1(N,h)=\mathcal{O}\Big(\sum_{2N-h<n<2N+h}|f(n)|\sum_{2N+h-n<a\le 2h}|f(n+a)|h\Big)=\mathcal{O}(h^3M_f^2).$

The sum on $s$ is not yet close to $W(a)$; hence, we transform $I_f(N,h)-D_f(N,h)$ into

$$2\sum_{N+h<n\le 2N-h} f(n)\sum_{0<a\le 2h} f(n+a)\sum_{\substack{|s|\le h\\ |s-a|\le h}} \mathrm{sgn}(s)\mathrm{sgn}(s-a)+R_1(N,h)+R_2(N,h),$$

say, with (again "tails") $R_2(N,h)=\mathcal{O}\Big(\sum_{N-h<n\le N+h}|f(n)|\sum_{0<a\le 2h}|f(n+a)|h\Big)=\mathcal{O}(h^3M_f^2).$

In order to get the right $n$-range

$$I_f(N,h)-D_f(N,h)=2\sum_{N<n\le 2N} f(n)\sum_{0<a\le 2h} f(n+a)\mathcal{G}(a)+R_1(N,h)+R_2(N,h)+R_3(N,h),$$

where, say, $\mathcal{G}(a)\overset{def}{=}\sum_{\substack{|s|\le h\\ |s-a|\le h}}\mathrm{sgn}(s)\mathrm{sgn}(s-a)\ll h,$ and therefore (still "tails")

$$R_3(N,h)=\mathcal{O}\Big(\Big(\sum_{N<n\le N+h}+\sum_{2N-h<n\le 2N}\Big)|f(n)|\sum_{0<a\le 2h}|f(n+a)|h\Big)=\mathcal{O}(h^3M_f^2).$$

An easy calculation shows that $\mathcal{G}(a)=W(a)+\mathcal{O}(1)$, whence $I_f(N,h)-D_f(N,h)$ is

$$2\sum_{N<n\le 2N} f(n)\sum_{0<a\le 2h} f(n+a)W(a)+R_1(N,h)+R_2(N,h)+R_3(N,h)+R_4(N,h),$$

with ("diagonal"-type remainders) $R_4(N,h)=\mathcal{O}\Big(\sum_{N<n\le 2N}|f(n)|\sum_{0<a\le 2h}|f(n+a)|\Big)=\mathcal{O}(NhM_f^2).$

Finally, being $W(a)$ even,

$$I_f(N,h)-D_f(N,h)=2\sum_{0<a\le 2h} W(a)\mathfrak{C}_f(a)+R_f(N,h),$$

$$R_f(N,h)=R_1(N,h)+R_2(N,h)+R_3(N,h)+R_4(N,h)\ll (Nh+h^3)M_f^2.\ \square$$

**Lemma 2.1.** *Let* $N,h\in\mathbb{N}$, *with* $h\to\infty$, $h=o(N)$ *as* $N\to\infty$. *Assume* $\exists\theta\in[0,1[$ *such that* $\sum_{d\le D}|g(d)|\ll D^{\theta}$ $(\forall D\ge 1)$ *and* $g(n)=0\ \forall n>3N$. *Then* $\forall\delta>0$ *(fixed) and any even function* $W(a)\ll h$ *with support in* $[-2h,2h]$

$$\sum_{a\neq 0}W(a)\left(\sum_{dq\le N}\sum g(d)g(q)\sum_{\substack{n\sim N\\ n\equiv 0(d)\\ n\equiv a(q)}}1\right)=N\sum_{a\neq 0}W(a)\sum_{(d,q)|a}\sum\frac{g(d)g(q)(d,q)}{dq}+\mathcal{O}\left(\frac{1}{1-\theta}N^{\theta+\delta}h^2\right).$$

*Proof.* First of all we remark that, writing $[d,q]=\frac{dq}{(d,q)}$ for the least common multiple of $d,q$, we have

$$\sum_{\substack{n\sim N\\ n\equiv 0(d)\\ n\equiv a(q)}}1=\frac{N}{[d,q]}+\mathcal{O}(1)=\frac{N(d,q)}{dq}+\mathcal{O}(1),\ \text{ if } (d,q)|a \text{ (otherwise the sum over } n \text{ is } 0\text{), whence}$$

$$\sum_{dq\le N}\sum g(d)g(q)\sum_{\substack{n\sim N\\ n\equiv 0(d)\\ n\equiv a(q)}}1=\sum_{\substack{dq\le N\\ (d,q)|a}}\sum g(d)g(q)\left(\frac{N(d,q)}{dq}+\mathcal{O}(1)\right)=$$

$$=N\sum_{(d,q)|a}\sum\frac{g(d)g(q)(d,q)}{dq}+\mathcal{O}\left(N\sum_{\substack{dq>N\\ (d,q)|a}}\sum\frac{|g(d)g(q)|(d,q)}{dq}\right)+\mathcal{O}\left(\sum_{\substack{dq\le N\\ (d,q)|a}}\sum|g(d)g(q)|\right);$$

due to $g(n)=0\ \forall n>3N$ we'll assume $d,q\le 3N$ in the remainders. The second one is negligible:

$$\sum_{a\neq 0}|W(a)|\sum_{\substack{dq\le N\\ (d,q)|a}}\sum|g(d)g(q)|\ll h\sum_{a\le 2h}\sum_{\ell|a}\sum_{\substack{dq\le N\\ (d,q)=\ell}}\sum|g(d)g(q)|$$

$$\ll h\sum_{\ell\le 2h}\left(\sum_{\substack{a\le 2h\\ a\equiv 0(\ell)}}1\right)\left(\sum_{d\le N}|g(d)|\sum_{q\le\frac{N}{d}}|g(q)|\right)\ll h\sum_{\ell\le 2h}\left(\frac{h}{\ell}+1\right)\sum_{d\le N}|g(d)|\left(\frac{N}{d}\right)^{\theta}\ll N^{\theta}h^2L^2$$

being in the final remainder. The hypothesis on $g$ together with the partial summation [T] gives

$$\sum_{d\le N}\frac{|g(d)|}{d^{\theta}}\ll 1+\int_1^N t^{\theta}\frac{dt}{t^{1+\theta}}\ll L,\qquad \sum_{\frac{N}{d}<q\le 3N}\frac{|g(q)|}{q}\ll N^{\theta-1}+\int_{\frac{N}{d}}^{3N}t^{\theta}\frac{dt}{t^2}\ll\frac{1}{1-\theta}\left(\frac{N}{d}\right)^{\theta-1},$$

the first used above and both to be applied (with the classical bound [H-W] $\delta>0\Rightarrow d(q)\ll q^{\delta}$) in

$$\sum_{a\neq 0}|W(a)|\left(N\sum_{\substack{dq>N\\ (d,q)|a}}\sum\frac{|g(d)g(q)|(d,q)}{dq}\right)\ll Nh\sum_{a\le 2h}\sum_{\ell|a}\ell\sum_{\substack{dq>N\\ (d,q)=\ell}}\sum\frac{|g(d)g(q)|}{dq}$$

$$\ll Nh\sum_{\ell\le 2h}\ell\left(\sum_{\substack{a\le 2h\\ a\equiv 0(\ell)}}1\right)\sum_{\substack{d,q\le 3N\\ dq>N\\ (d,q)=\ell}}\sum\frac{|g(d)g(q)|}{dq}\ll Nh^2\sum_{\ell\le 2h}\sum_{\substack{d,q\le 3N\\ dq>N\\ (d,q)=\ell}}\sum\frac{|g(d)g(q)|}{dq}$$

$$\ll Nh^2\sum_{\substack{d,q\le 3N\\ dq>N}}\sum\frac{|g(d)g(q)|}{dq}d(q)\ll N^{\delta}Nh^2\sum_{d\le 3N}\frac{|g(d)|}{d}\sum_{\frac{N}{d}<q\le 3N}\frac{|g(q)|}{q}\ll\frac{N^{\theta+\delta}h^2L}{1-\theta}.\ \square$$

**Lemma 2.2.** *Let* $N, h \in \mathbb{N}$, *with* $h \to \infty$, $h = o(N)$ *as* $N \to \infty$. *Assume that* $\forall \delta > 0$ *(fixed)* $g(n) \ll n^\delta$, *that* $\exists \theta \in [0,1[\, : \sum_{d \le D} |g(d)| \ll D^\theta$ $(\forall D \ge 1)$ *and* $g(n) = 0$ $\forall n > 3N$. *Then for any even function* $W(a) \ll h$ *with support in* $[-2h, 2h]$

$$\sum_{a \neq 0} W(a) \left( \sum_{dq > N} \sum g(d) g(q) \sum_{\substack{n \sim N \\ n \equiv 0(d) \\ n \equiv a(q)}} 1 \right) \ll \frac{N^{\frac{2\theta}{1+\theta} + \delta} h^2}{1 - \theta}.$$

*Proof.* First of all we split our double sum over $d$ and $q$ into, say

$$S = S(a, N) \overset{def}{=} \sum_{\substack{d \\ dq > N}} \sum_{q} g(d) g(q) \sum_{\substack{n \sim N \\ n \equiv 0(d) \\ n \equiv a(q)}} 1 \ll S_1 + S_2 + S_3,$$

where

$$S_1 = S_1(a, N) \overset{def}{=} \sum_{\substack{d, q \le N^\beta \\ dq > N}} \sum |g(d) g(q)| \sum_{\substack{n \sim N \\ n \equiv 0(d) \\ n \equiv a(q)}} 1 \ll N \sum_{\substack{d,\, q \le N^\beta \\ dq > N \\ (d,q) | a}} \sum \frac{|g(d)||g(q)|}{[d, q]} + \sum_{\substack{d,\, q \le N^\beta \\ dq > N}} \sum |g(d)||g(q)|$$

$$\ll N \sum_{\ell | a} \ell \sum_{d \le N^\beta} \frac{|g(d)|}{d} \sum_{\substack{\frac{N}{d} < q \le N^\beta \\ (q,d) = \ell}} \frac{|g(q)|}{q} + N^{2\theta\beta}$$

is "of type I"(see the previous Lemma for the bound over the $n-$sum), and

$$S_2 = S_2(a, N) \overset{def}{=} \sum_{\substack{d > N^\beta \\ dq > N}} \sum |g(d) g(q)| \sum_{\substack{n \sim N \\ n \equiv 0(d) \\ n \equiv a(q)}} 1, \qquad S_3 = S_3(a, N) \overset{def}{=} \sum_{\substack{q > N^\beta \\ dq > N}} \sum |g(d) g(q)| \sum_{\substack{n \sim N \\ n \equiv 0(d) \\ n \equiv a(q)}} 1$$

are of type II; here $\beta \in ]0, 1[$ will be chosen later. We have ($\delta$'s not the same at each occurrence)

$$S_2 \ll N^\delta \sum_{N^\beta < d \le 3N} |g(d)| \sum_{\substack{n \sim N \\ n \equiv 0(d)}} d(n - a) \ll N^\delta \left( N \sum_{N^\beta < d \le 3N} \frac{|g(d)|}{d} + \sum_{N^\beta < d \le 3N} |g(d)| \right)$$

and similarly for the other of type II. The estimates

$$\sum_{N^\beta < d \le 3N} \frac{|g(d)|}{d} \ll \frac{N^{\beta(\theta - 1)}}{1 - \theta} \quad \text{and} \quad \sum_{\frac{N}{d} < q \le N^\beta} \frac{|g(q)|}{q} \ll \frac{1}{1 - \theta} \left( \frac{N}{d} \right)^{\theta - 1}, \quad \sum_{d \le 3N} \frac{|g(d)|}{d^\theta} \ll L$$

follow from applying the hypothesis on $|g|-$sum and partial summation. These (with $\beta < 1, \theta < 1$) give

$$S_2 + S_3 \ll_\theta N^\delta (N^{1 + \beta(\theta - 1)} + N^\theta) \ll_\theta N^{1 + \beta(\theta - 1) + \delta} \;\Rightarrow\; \sum_{a \neq 0} W(a)(S_2 + S_3) \ll_\theta N^{1 + \beta(\theta - 1) + \delta} h^2,$$

$$\sum_{a \neq 0} W(a) S_1 \ll N h \sum_{\ell \le 2h} \ell \sum_{d \le N^\beta} \frac{|g(d)|}{d} \sum_{\substack{\frac{N}{d} < q \le N^\beta \\ (q,d) = \ell}} \frac{|g(q)|}{q} \sum_{\substack{a \le 2h \\ a \equiv 0(\ell)}} 1 + N^{2\theta\beta} h^2 \ll_\theta \left( N^\theta L + N^{2\theta\beta} \right) h^2,$$

abbreviating $\ll_\theta$ to mean $\frac{1}{1-\theta}$ in the constant; whence, optimally choosing $\beta = \frac{1}{1+\theta}$, we finally get

$$\sum_{a \neq 0} W(a) S \ll \frac{N^{\frac{2\theta}{1+\theta} + \delta} h^2}{1 - \theta}. \quad \square$$

## 3. Proof of the Theorem.

First of all, applying Lemma 2.0, we write the symmetry integral in terms of the correlations, i.e. $\mathcal{C}_f(a)$, say

$$\mathcal{C}_f(a) = \sum_{n\sim N} f(n)f(n-a) = \sum_{n\sim N} (g*1)(n)(g*1)(n-a) = \sum_{n\sim N} \sum_{d|n} g(d) \sum_{q|n-a} g(q) = \sum_{d} g(d) \sum_{q} g(q) \sum_{\substack{n\sim N \\ n\equiv 0(d) \\ n\equiv a(q)}} 1$$

(here the term with $a = 0$ gives the first term in RHS of the Theorem, since $W(0) = 2h$) and then apply Lemmas 2.1 and 2.2, after having observed that

$$\mathcal{O}\left(\left(Nh + h^3\right) M_f^2\right) = \mathcal{O}\left(\left(Nh + h^3\right) N^{\delta}\right)$$

since our hypothesis on $g$ is (not the same $\delta$ at each occurrence)

$$g(n) \ll n^{\delta} \quad \Rightarrow \quad f(n) \ll n^{\delta} \quad \Rightarrow \quad M_f^2 \ll N^{\delta} \quad \forall \delta > 0. \ \square$$

## 4. Some remarks for the Theorem.

First of all, we notice that the double sum (over $d, q$) in the main term of the symmetry integral, $I_f(N, h)$, appearing in our Theorem is absolutely convergent, being so each series (both over $d$ and $q$), since partial summation gives

$$\sum_{d\le D} |g(d)| \ll D^{\theta} \ (\theta < 1) \ \Rightarrow \ \sum_{d=1}^{\infty} \frac{|g(d)|}{d} < \infty$$

and in a moment we will express the main term of $I_f(N, h)$ in a more manageable form.

Let us mention, before, that much of our inspiration for the Lemmas 2.1 and 2.2 (which constitute the bulk of the paper) comes from a 1949 work of Mirsky ("Summation formulas for arithmetical functions"), from which we draw the main ideas in proving Lemma 2.2.

Our main concern, now, is to express the main term of our symmetry integral through a kind of "singular series" which is similar to the one appearing in our previous paper (namely, II).

In order to perform this link, we need to remark that in the proof of Lemma 0 in II a result of Wintner [W] allowed (and still does, in our hypotheses on $g$) to write, for the (Fourier-)Ramanujan coefficients, say $a_q(g*\mathbf{1})$, of the arithmetical function $f = g*\mathbf{1}$ :

$$a_q(g*\mathbf{1}) = \sum_{n\equiv 0(q)} \frac{g(n)}{n}$$

(here with $n \equiv 0(q)$ we mean the summation over all natural $n$ multiples of $q$), whence, say,

$$\sum_{(d,q)|a}\sum \frac{g(d)g(q)}{[d,q]} = \sum_{(d,q)|a}\sum \frac{g(d)g(q)}{d\,q}(d,q) = \sum_{\ell|a} \ell \sum_{(d,q)=\ell}\sum \frac{g(d)}{d}\frac{g(q)}{q} = \sum_{\ell|a} \ell \sum_{(m,n)=1}\sum \frac{g(\ell m)}{\ell m}\frac{g(\ell n)}{\ell n} =$$

$$= \sum_{\ell|a} \ell \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{g(\ell m)}{\ell m}\frac{g(\ell n)}{\ell n} \sum_{j|(m,n)} \mu(j) = \sum_{\ell|a} \ell \sum_{j=1}^{\infty} \mu(j) a_{\ell j}^2(g*\mathbf{1}) = \sum_{q=1}^{\infty} a_q^2(g*\mathbf{1}) \sum_{\substack{\ell|a \\ \ell|q}} \ell\mu\left(\frac{q}{\ell}\right) =$$

$$= \sum_{q=1}^{\infty} a_q^2(g*\mathbf{1}) c_q(a) = \sum_{q=1}^{\infty} \left|a_q(g*\mathbf{1})\right|^2 c_q(a),$$

being $f$, whence $g$ and then $a_q(g*\mathbf{1})$ real; here

$$c_q(a) \overset{def}{=} \sum_{n\in\mathbb{Z}_q^*} e_q(an) \qquad (e_q(r) \overset{def}{=} e^{2\pi i r/q})$$

is the classical **Ramanujan sum**, for which we have used Hölder's formula

$$c_q(a) = \sum_{\substack{\ell|a\\ \ell|q}} \ell\mu\left(\frac{q}{\ell}\right).$$

Hence, we are entitled to compare our present "singular series" i.e.

$$\mathfrak{S}_f(h) \overset{def}{=} \sum_{a\neq 0} W(a) \sum_{q=1}^{\infty} |a_q(g*\mathbf{1})|^2 c_q(a),$$

with the one first seen in our previous work, II, i.e.

$$\mathfrak{S}_{g*\mathbf{1}}(h) = \sum_{q\le h} |a_q(g*\mathbf{1})|^2 \sum_k W(k)c_q(k).$$

Thus, our Theorem amounts (in the right hypotheses, see before) to the formula

$$I_f(N,h) = \mathfrak{S}_f(h)N + R_\theta(N,h),$$

where the remainder is the same as above, i.e.

$$R_\theta(N,h) \ll \frac{1}{1-\theta}\left(\left(N^{\frac{2\theta}{1+\theta}}h + N + h^2\right)N^\delta h\right).$$

We remark that the first term on the RHS of the Theorem is bounded as (not the same $\delta$ at each occurrence)

$$2h \sum\sum_{d,q\,\le\,2N+h} g(d)g(q) \sum_{\substack{n\sim N\\ n\equiv 0([d,q])}} 1 \ll h\sum_{n\sim N}\sum_{d|n}\sum_{q|n} N^{2\delta} \ll NhN^\delta$$

still from the classical bound for the divisor function. Then, its inclusion in our "singular series" would not affect the result.

Thus, the term $k=0$ is not really a difference between the two series.
However, when compared to the one appearing in our previous result it also has more terms (tails, for $q>h$); nonetheless, we trust the hypothesis on $g$ to render these contributes unimportant for the asymptotics of the two singular series. Also, if $g$ satisfies the hypothesis of II we get from Lemma 0 of II : $a_q(g*\mathbf{1}) \ll_\varepsilon 1/q^{1+\varepsilon}$, whence Lemma 3 of II (giving $\left|\sum_{a\equiv 0(d)} W(a)\right| \ll h$, $\forall d\in\mathbb{N}$) lets us estimate the "tails" as follows:

$$\sum_a W(a)c_q(a) = \sum_{d|q} d\mu\left(\frac{q}{d}\right) \sum_{a\equiv 0(d)} W(a) \ll qhd(q) \;\Rightarrow\; \sum_{q>h} |a_q(f)|^2 \sum_a W(a)c_q(a) \ll_\varepsilon h\sum_{q>h}\frac{d(q)}{q^{1+2\varepsilon}} \ll_\varepsilon \frac{h}{h^{2\varepsilon-\delta}}$$

that points in the right direction : tails are "small" !

(Notwithstanding supports of evidence, we will not give a proof of the fact that, in one of the two hypotheses for $g$, the present or the other of II, we have an asymptotic relation among the respective singular series !)

## References


**[C1]** Coppola, G.- *On the symmetry of divisor sums functions in almost all short intervals* - INTEGERS **4** (2004), A2, 9pp. (electronic). MR 2005b:11153

**[C2]** Coppola, G.- *On the symmetry of square-free numbers in almost all short intervals* - Riv. Mat. Univ. Parma (Serie 7) **3*** (2004), 205-212. MR 2005k:11192

**[C3]** Coppola, G.- *On the symmetry of arithmetical functions in almost all short intervals, II* - to appear.

**[C-S]** Coppola, G. and Salerno, S.- *On the symmetry of arithmetical functions in almost all short intervals* - C. R. Math. Rep. Acad. Sci. Canada **26, no.4** (2004), 118-125. MR 2005m:11180

**[D]** Davenport, H.- *Multiplicative Number Theory* - Third Edition, GTM 74, Springer, New York, 2000. MR 2001f:11001

**[H-W]** Hardy, G.H. and Wright, E.M.- *An Introduction to the Theory of Numbers* - Fifth Edition. The Clarendon Press, Oxford University Press, New York, 1979. MR 81i:10002

**[T]** Tenenbaum, G. - *Introduction to analytic and probabilistic number theory* - Cambridge studies in advanced mathematics : 46, Cambridge University Press, 1995. MR 97e:11005b

**[W]** Wintner, A. - *Eratosthenian Averages* - Baltimore Md., 1943. MR 7,366a



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