ON THE SYMMETRY OF ARITHMETICAL FUNCTIONS IN ALMOST ALL SHORT INTERVALS, II

by

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1. Introduction and statement of the results.

We are interested, as in our previous paper [C-S] (which we'll quote sometimes as I), in studying the symmetry of some arithmetical functions in almost all short intervals (say, $[x-h, x+h]$). Unlike in our previous paper, we do not seek functions f whose behaviour in the short intervals can be studied through an immediate application of the Large Sieve inequality (since their "Eratosthenes transform " $f * \mu$ is supported up to, substantially, $\sqrt{x}\,\sqrt{h}$, see Theorem 1 [C-S]).

Instead, we study another class (not disjoint to the one quoted) of f, for which $(f*\mu)(n) \ll n^{-\varepsilon}$ with an ε (strictly !) positive. For these functions we apply a Large Sieve "combined" with Ramanujan expansions. The Large Sieve alone gives (sometimes, strictly) weaker results for this class of functions.

In fact, the strange phenomenon we point out is that, whenever $f = g * \mathbf{1}$ (hence calling, now on, g the Eratosthenes transform of f), the Large Sieve "alone" is unable to give good results if $g(n) \ll n^{-\varepsilon}$, with $\varepsilon \in]0, 1/4[$, but, in the same case, the combined-Large Sieve gives non-trivial bounds.

Also, whenever $g(n) \ll n^{-\varepsilon}$ and $(0<)\varepsilon < 1$ our method works better.

Thus we get further cancellation from the application of the Ramanujan expansions.

This is possible, partly due to the fact that, in our functions f, the Eratosthenes transform g has bounded support (depending on N, h, namely $g(n) = 0\ \forall n > 2N + h$, see after). This (see Lemma 0) renders our Ramanujan(-Fourier) expansions ALL absolutely convergent (hence, much easier to handle). Also, and this is the key-point, the hypothesis $g(n) \ll n^{-\varepsilon}$ turns the Ramanujan expansion "tails" into a kind of "arithmetical smoothing"(clear in Lemma 1).

We'll give bounds on the symmetry of our f, see Theorem 1, but, also, in order to give explicit results (though apparently weaker), we'll supply an "asymptotic" result, namely Theorem 2.

First of all, we assume throughout that $N, h \in \mathbb{N}$, with $h = h(N) \to \infty$ and (in order to get short intervals) $h = o(N)$ (even if, with some care, our results may hold for $h \le cN$, $0 < c < 1$ suitable).

Then, we'll write $x \sim N$ to mean: $N < x \le 2N$, for x an integer variable.

We can define, for such x, the **symmetry sum** of (the arithmetical function) f to be:

$$S_f^{\pm}(x) = S_f^{\pm}(x,h) \stackrel{def}{=} \sum_{|n-x|\le h} \mathrm{sgn}(n-x) f(n),$$

where $\mathrm{sgn}(r)$ is the sign of $r \in \mathbb{R}$, i.e. $\mathrm{sgn}(r) \stackrel{def}{=} r/|r|\ \forall r \neq 0$, $\mathrm{sgn}(0) \stackrel{def}{=} 0$.

We are interested in its (discrete-)mean-square, the **symmetry integral** (for analogy, see [K-P], with the Selberg integral, but for technical reasons it's a finite sum) of f, namely:

$$I_f(N,h) \stackrel{def}{=} \sum_{x\sim N} \left| \sum_{|n-x|\le h} \mathrm{sgn}(n-x) f(n) \right|^2,$$

whence, assuming (as we'll do now on) that $f = g * \mathbf{1}$,

$$I_f(N,h) = I_{g*\mathbf{1}}(N,h) = \sum_{x\sim N} \left| \sum_{|n-x|\le h} \mathrm{sgn}(n-x) \sum_{d|n} g(d) \right|^2 \quad \text{(Henceforth assume } g(n)=0, \forall n > 2N+h).$$

This integral checks the symmetry of f in "almost all" short intervals $[x-h, x+h]$ (with $N < x \le 2N$), i.e. in all of them, with possibly $o(N)$ exceptions.

Here we set $L \stackrel{def}{=} \log N$.

We give, now, our first result:

Theorem 1. *Let* $N, h \in \mathbb{N}$, $h \to \infty$, $h = o(N)$ *as* $N \to \infty$. *Let* $g : \mathbb{N} \to \mathbb{C}$ *be an arithmetical function. Fix (a real number)* $\varepsilon > 0$. *Then*

$$g(n) \ll \frac{1}{n^{\varepsilon}} \quad \Rightarrow \quad I_{g*\mathbf{1}}(N,h) \ll_{\varepsilon} Nh^2 \left(\frac{L^2}{h} + \frac{L^{3-\varepsilon}}{N^{\varepsilon} h^{\varepsilon}} \right),$$

where the $\ll$*-constant, say* $c(\varepsilon)$, *depends only on* ε *(namely, can take* $c(\varepsilon) = (1 + \frac{1}{\varepsilon})^4$*).*

As an immediate consequence, we prove (choosing h & ε properly) our

Corollary 1. *Let* $N, h \in \mathbb{N}$, *with* $N^{\theta} \ll h \ll N^{\theta}$, *for some* $0 < \theta < 1$. *Let* $g : \mathbb{N} \to \mathbb{C}$ *be an arithmetical function. Fix (a real number)* $\varepsilon > \frac{\theta}{1+\theta}$. *Then*

$$g(n) \ll \frac{1}{n^{\varepsilon}} \quad \Rightarrow \quad I_{g*\mathbf{1}}(N,h) \ll_{\varepsilon} NhL^2,$$

with, again, the same $c(\varepsilon)$.

Also, we can combine Ramanujan expansions and the Large Sieve for "small" moduli q (say, $q \ll h$) to get an asymptotic result:

Theorem 2. *Let* $N, h \in \mathbb{N}$, $h \to \infty$, $h = o(N)$ *as* $N \to \infty$. *Let* $g : \mathbb{N} \to \mathbb{C}$ *be an arithmetical function. Fix (a real number)* $\varepsilon > 0$. *Then*

$$g(n) \ll \frac{1}{n^{\varepsilon}} \quad \Rightarrow \quad I_{g*\mathbf{1}}(N,h) = \mathfrak{S}_{g*\mathbf{1}}(h)N + \mathcal{O}_{\varepsilon}\left(Nh^2 \frac{L^{5/2}}{h^{\varepsilon}} \left(\frac{1}{\sqrt{h}} + \frac{h}{\sqrt{N}} + \frac{\sqrt{L}}{h^{\varepsilon}} \right) \right),$$

with a main term (say, $c_q(n) \stackrel{def}{=} \sum_{r \in \mathbb{Z}_q^*} e^{2\pi i r n/q}$, *now on, is the classical Ramanujan sum)*

$$\mathfrak{S}_{g*\mathbf{1}}(h) \stackrel{def}{=} \sum_{q \le h} |a_q(g*\mathbf{1})|^2 \sum_k W(k) c_q(k)$$

where $W(k)$ *is defined in Lemma 3 and the* $\ll$*-const. depends only on* ε *(e.g., same* $c(\varepsilon)$ *as above).*

(In the sequel, we'll abbreviate $n \equiv r(\bmod q)$ with $n \equiv r(q)$.)

Using Lemma 0, with $c_q(k)$ as in Lemma 1 & $\left| \sum_{k \equiv 0(d)} W(k) \right| \ll h$ (see Lemma 3), the "singular series" is

$$\mathfrak{S}_{g*\mathbf{1}}(h) \ll \left(1 + \frac{1}{\varepsilon}\right)^2 \sum_{q \le h} \frac{1}{q^{2+2\varepsilon}} \sum_{d|q} d \left| \sum_{\substack{k \\ k \equiv 0(d)}} W(k) \right| \ll \left(1 + \frac{1}{\varepsilon}\right)^2 h \sum_{d \le h} \frac{1}{d^{1+2\varepsilon}} \sum_{j \le \frac{h}{d}} \frac{1}{j^{2+2\varepsilon}} \ll \left(1 + \frac{1}{\varepsilon}\right)^3 h.$$

Then, we prove (choosing h "not too small") our

Corollary 2. *Let* $N, h \in \mathbb{N}$, *and* $N^{\theta} \ll h \ll N^{\theta}$, *with* $0 < \theta < 1/2$. *Let* $g : \mathbb{N} \to \mathbb{C}$ *and fix (a real number)* $\varepsilon > 0$. *Then, for a suitable* $\alpha > 0$ *(depending on* ε, θ*)*

$$g(n) \ll \frac{1}{n^{\varepsilon}} \quad \Rightarrow \quad I_{g*\mathbf{1}}(N,h) = \mathfrak{S}_{g*\mathbf{1}}(h)N + \mathcal{O}\left(Nh^2 N^{-\alpha}\right).$$

2. Lemmas.

Our first Lemma asserts that

$$g(n) \ll_{\varepsilon} \frac{1}{n^{\varepsilon}} \Leftrightarrow a_q(g * \mathbf{1}) \ll_{\varepsilon} \frac{1}{q^{1+\varepsilon}},$$

indicating with $a_q(f)$ the (Fourier-)Ramanujan q-th coefficient of f; more precisely:

Lemma 0. *Let $N, h \in \mathbb{N}$, with $h \to \infty$, $h = o(N)$ as $N \to \infty$. Let $g : \mathbb{N} \to \mathbb{C}$ have finite support (depending on N, h). Then $\forall \varepsilon > 0$*

$$|g(n)| \le \frac{c}{n^{\varepsilon}} \ \ \forall n \in \mathbb{N} \ \ \Rightarrow \ \ |a_q(g * \mathbf{1})| \le \left(1 + \frac{1}{\varepsilon}\right) \frac{c}{q^{1+\varepsilon}} \ \ \forall q \in \mathbb{N}$$

and

$$|a_q(g * \mathbf{1})| \le \frac{c}{q^{1+\varepsilon}} \ \ \forall q \in \mathbb{N} \ \ \Rightarrow \ \ |g(n)| \le \left(1 + \frac{1}{\varepsilon}\right) \frac{c}{n^{\varepsilon}} \ \ \forall n \in \mathbb{N}$$

where in both cases $c > 0$ is an absolute constant.

Proof. Since $g(n) = 0$, $\forall n > 2N + h \Rightarrow$ the series in Wintner's Theorem [W] are absolutely convergent,

$$a_q(g * \mathbf{1}) = \sum_{\substack{n=1 \\ n \equiv 0(q)}}^{\infty} \frac{g(n)}{n} \Leftrightarrow g(d) = d \sum_{j=1}^{\infty} \mu(j) a_{dj}(g * \mathbf{1}), \quad \text{a kind of Möbius inversion formula.}$$

The first part comes from the formula on the left &, resp., the second part from that on the right:

$$|g(n)| \le \frac{c}{n^{\varepsilon}} \ \ \Rightarrow \ \ |a_q(g * \mathbf{1})| \le \sum_{\substack{n=1 \\ n \equiv 0(q)}}^{\infty} \frac{|g(n)|}{n} \le \frac{c}{q^{1+\varepsilon}} \sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} \le \frac{c}{q^{1+\varepsilon}} \left(1 + \int_1^{\infty} \frac{dx}{x^{1+\varepsilon}}\right) \ \ \&$$

$$|a_q(g * \mathbf{1})| \le \frac{c}{q^{1+\varepsilon}} \ \ \Rightarrow \ \ |g(d)| \le d \sum_{j=1}^{\infty} |a_{dj}(g * \mathbf{1})| \le \frac{c}{d^{\varepsilon}} \sum_{j=1}^{\infty} \frac{1}{j^{1+\varepsilon}} \le \frac{c}{d^{\varepsilon}} \left(1 + \int_1^{\infty} \frac{dx}{x^{1+\varepsilon}}\right). \ \square$$

Lemma 1. *Let $N, h \in \mathbb{N}$, with $h \to \infty$, $h = o(N)$ as $N \to \infty$ and $\forall \varepsilon > 0$ let $K = K(\varepsilon) > 0$. Then*

$$a_q \ll \frac{K}{q^{1+\varepsilon}}, \ Q > 1 \ \ \Rightarrow \ \ \sum_{x \sim N} \left| \sum_{q > Q} a_q \sum_{|n-x| \le h} c_q(n) \mathrm{sgn}(n-x) \right|^2 \ll \left(1 + \frac{1}{\varepsilon}\right)^2 \frac{K^2 L^3}{Q^{2\varepsilon}} N h^2.$$

Proof. Set $\Sigma = \sum_{q>Q} a_q \sum_{|n-x| \le h} c_q(n) \mathrm{sgn}(n-x)$ and use $c_q(n) = \sum_{d|q, d|n} d\mu(\frac{q}{d})$ [D,ch.26] to get

$$\chi_d(x) = \sum_{|n-x| \le h, n \equiv 0 (\mathrm{mod}\ d)} \mathrm{sgn}(n-x) \qquad \Rightarrow \qquad \Sigma = \sum_{d \le 2N+h} d \chi_d(x) \sum_{j > Q/d} \mu(j) a_{dj}$$

($d > 2N + h \ge x + h \Rightarrow \chi_d(x) = 0$; $\chi_d(x)$ isn't a Dirichlet character !). The hypothesis on a_q gives

$$\Sigma \ll K \left(\frac{1}{\varepsilon} \sum_{d \le Q} \frac{|\chi_d(x)|}{d^{\varepsilon}} \left(\frac{d}{Q}\right)^{\varepsilon} + \left(1 + \frac{1}{\varepsilon}\right) \sum_{d > Q} \frac{|\chi_d(x)|}{d^{\varepsilon}} \right) \ll \left(1 + \frac{1}{\varepsilon}\right) \frac{K}{Q^{\varepsilon}} \sum_{d \le 2N+h} |\chi_d(x)|.$$

Since $|\chi_d(x)| \le \sum_{|m - \frac{x}{d}| \le \frac{h}{d}} 1 \ \ \Rightarrow \ \ \sum_d |\chi_d(x)| \ll \sum_{|n-x| \le h} d(n)$, apply Cauchy inequality, exchange the x, n-sums and use the classical estimate [H-W] $\sum_{n \le X} d^2(n) \ll X \log^3 X$ to get the bound:

$$\left(1 + \frac{1}{\varepsilon}\right)^{-2} \sum_{x \sim N} |\Sigma|^2 \ll \frac{K^2}{Q^{2\varepsilon}} \sum_{x \sim N} \left(\sum_{|n-x| \le h} d(n) \right)^2 \ll \frac{K^2 h^2}{Q^{2\varepsilon}} \sum_{N-h < n \le 2N+h} d^2(n) \ll \frac{K^2 N h^2 L^3}{Q^{2\varepsilon}}. \ \square$$

Lemma 2. *Let $N, h \in \mathbb{N}$, with $h \to \infty$, $h = o(N)$ as $N \to \infty$ and $\forall \varepsilon > 0$ let $K = K(\varepsilon) > 0$. Then*

$$a_q \ll \frac{K}{q^{1+\varepsilon}},\ D > 1 \ \Rightarrow \ \sum_{x\sim N} \left| \sum_{q\sim D} a_q \sum_{|n-x|\le h} c_q(n)\mathrm{sgn}(n-x) \right|^2 \ll K^2(N+D^2)hD^{-2\varepsilon}.$$

Proof. First, recall Ramanujan sum definition $c_q(n) = \sum_{j\le q}^{\ *} e_q(nj)$ (j's in the reduced classes (mod q)):

$$\sum_{q\sim D} a_q \sum_{|n-x|\le h} c_q(n)\mathrm{sgn}(n-x) = \sum_{q\sim D} a_q \sum_{j\le q}{}^{*} \Big(\sum_{|s|\le h (\mathrm{mod}\ q)} e_q(js)\mathrm{sgn}(s) \Big) e_q(jx),$$

since the orthogonality of additive characters ([V] or [D,ch.25]) allows us to substitute $h(\mathrm{mod}\ q)$ to h. Then, the Large Sieve inequality [C-S] and a_q estimate give (at last, $h(\mathrm{mod}\ q) \le h$)

$$\sum_{x\sim N} \Big| \sum_{q\sim D} a_q \sum_{|n-x|\le h} c_q(n)\mathrm{sgn}(n-x) \Big|^2 \ll \sum_{x\sim N} \Big| \sum_{q\sim D} a_q \sum_{j\le q}{}^{*} \Big(\sum_{s\le h(\mathrm{mod}\ q)} \sin\Big(\frac{2\pi js}{q}\Big)\Big) e_q(jx) \Big|^2$$

$$\ll (N+D^2) \sum_{q\sim D} |a_q|^2 \sum_{j\le q} \Big| \sum_{s\le h(\mathrm{mod}\ q)} \sin\Big(\frac{2\pi js}{q}\Big) \Big|^2 \ll K^2(N+D^2)h \sum_{q\sim D} \frac{1}{q^{1+2\varepsilon}}. \ \square$$

Lemma 3. *Let $N, h \in \mathbb{N}$, with $h \to \infty$, $h = o(N)$ as $N \to \infty$. Then, for each sequence of numbers $a_q \in \mathbb{C}$*

$$J \to \infty, J \ll h \ \Rightarrow \ \sum_{x\sim N} \left| \sum_{q\le J} a_q \sum_{|n-x|\le h} c_q(n)\mathrm{sgn}(n-x) \right|^2 = \sum_{q\,,\,r\,\le J}\sum a_q \overline{a_r} \sum_k W(k) \sum_{n\sim N} c_q(n)c_r(n-k)$$

$$+\mathcal{O}\left((Nh+h^3) \left(\sum_{q\le J} d(q)|a_q| \right)^2 \right),$$

where the weight $W(a)$ is even and defined on non-negative integers as

$$W(k) \stackrel{def}{=} \begin{cases} 2h-3k & \text{if } 0 \le k \le h \\ k-2h & \text{if } h \le k \le 2h \\ 0 & \text{otherwise} \end{cases},$$

whence it has zero mean-value (over all the integers k); also, $\left|\sum_{k\equiv 0(d)} W(k)\right| \ll h$, uniformly $\forall d \in \mathbb{N}$.

Proof. We expand the square (recall that $c_q(n) \in \mathbb{R}$) and exchange the sums in LHS, say $\Delta_J = \Delta_J(N,h,f)$,

$$\Delta_J = \sum_{q\,,\,r\,\le J}\sum a_q \overline{a_r} \sum_{x\sim N} \sum_{|n_1-x|\le h} \sum_{|n_2-x|\le h} c_q(n_1)c_r(n_2)\mathrm{sgn}(n_1-x)\mathrm{sgn}(n_2-x).$$

This is a "no-main-term dispersion method". In fact, exchanging the inner sums, say

$$S_{q,r} \stackrel{def}{=} \sum_{x\sim N} \sum_{|n_1-x|\le h} \sum_{|n_2-x|\le h} c_q(n_1)c_r(n_2)\mathrm{sgn}(n_1-x)\mathrm{sgn}(n_2-x)$$

$$= \sum_{N-h<n_1,n_2\le 2N+h}\sum c_q(n_1)c_r(n_2) \sum_{\substack{x\sim N \\ |x-n_1|\le h \\ |x-n_2|\le h}} \mathrm{sgn}(x-n_1)\mathrm{sgn}(x-n_2).$$

Clearly, the inner sum is empty when $|n_1 - n_2| > 2h$. Now on we "clear-up" the main term:

$$S_{q,r} = \sum_{\substack{N+h<n_1,n_2\le 2N-h\\|n_1-n_2|\le 2h}}\sum c_q(n_1)c_r(n_2)\sum_{\substack{|x-n_1|\le h\\|x-n_2|\le h}}\mathrm{sgn}(x-n_1)\mathrm{sgn}(x-n_2) + \mathcal{O}\left(h\mathcal{E}_q(2h)\mathcal{E}_r(2h)\right),$$

say, defining the useful quantity $\mathcal{E}_q(M) \stackrel{def}{=} d(q)(M+q)$, $\forall M \ge 1$, since we need, uniformly $\forall X \in \mathbb{Z}$,

$$\sum_{A+X<n\le B+X}|c_q(n)| \ll \sum_{A+X<n\le B+X}\sum_{\substack{d|q\\d|n}} d \ll \sum_{d|q}(B-A+d) \ll d(q)(B-A+q) = \mathcal{E}_q(B-A).$$

Due to our hypotheses we easily get, say,

$$J \ll h \;\Rightarrow\; \mathcal{E}_q(2h) \ll hd(q) \;\Rightarrow\; R_1(q,r) \stackrel{def}{=} \mathcal{O}\Big(h\mathcal{E}_q(2h)\mathcal{E}_r(2h)\Big) = \mathcal{O}\Big(h^3 d(q)d(r)\Big).$$

We see that it contributes to Δ_J remainders as h^3 times the square of the sum over q.
Let's change variables to let the weight $W(a)$ appear; set $n = n_1$, $a = n_2 - n_1$ in

$$S_{q,r} = \sum_{N+h<n\le 2N-h}\ \sum_{\substack{0\le|a|\le 2h\\N+h-n<a\le 2N-h-n}} c_q(n)c_r(n+a)\mathcal{G}(a) + R_1(q,r),$$

say, where an easy calculation shows that, say,

$$\mathcal{G}(a) \stackrel{def}{=} \sum_{\substack{|x-n|\le h\\|x-n-a|\le h}}\mathrm{sgn}(x-n)\mathrm{sgn}(x-n-a) = \sum_{\substack{|s|\le h\\|s-a|\le h}}\mathrm{sgn}(s)\mathrm{sgn}(s-a) = W(a) + \mathcal{O}(1).$$

Thus

$$S_{q,r} = \sum_{N+h<n\le 2N-h}\ \sum_{\substack{0\le|a|\le 2h\\N+h-n<a\le 2N-h-n}} c_q(n)c_r(n+a)W(a) + R_1(q,r) + R_2(q,r),$$

where, say

$$R_2(q,r) \stackrel{def}{=} \mathcal{O}\left(\sum_{N+h<n\le 2N-h}|c_q(n)|\sum_{0\le|a|\le 2h}|c_r(n+a)|\right) = \mathcal{O}(\mathcal{E}_q(N)\mathcal{E}_r(2h));$$

using our hypotheses (see above)

$$J \ll h,\ h = o(N) \;\Rightarrow\; \mathcal{E}_q(N) \ll Nd(q) \;\Rightarrow\; R_2(q,r) = \mathcal{O}\Big(Nhd(q)d(r)\Big).$$

This gives another contribute to Δ_J remainders: Nh times the square of the sum over q.
We simplify in two steps the sum over a:

$$S_{q,r} = \sum_{N+3h<n\le 2N-3h}\ \sum_{0\le|a|\le 2h} c_q(n)c_r(n+a)W(a) + \mathcal{O}\left(R_1(q,r) + R_2(q,r)\right)$$

$$= \sum_a W(a)\sum_{n\sim N} c_q(n)c_r(n+a) + \mathcal{O}\left(R_1(q,r) + R_2(q,r)\right),$$

making in both "invisible" errors, because as big as $R_1(q,r)$. Finally, being $W(a)$ even,

$$S_{q,r} = \sum_a W(a)\sum_{n\sim N} c_q(n)c_r(n-a) + \mathcal{O}\left(R_1(q,r) + R_2(q,r)\right). \;\square$$

Lemma 4. *Let* $N, q, r \in \mathbb{N}$ *and* $a \in \mathbb{Z}$. *Then, as* $N \to \infty$

$$\sum_{n \sim N} c_q(n) c_r(n-a) = \begin{cases} c_q(a)N + \mathcal{O}(q^2 \log(q+1)) & \text{if } q = r \\ \mathcal{O}\left(qr \log(q+1)\log(r+1)\right) & \text{if } q \neq r \end{cases} .$$

Proof. First, the case $q = r = 1$: $c_1(m) = 1 \ \forall m \in \mathbb{Z} \Rightarrow$ the correlation (i.e., LHS) is $c_1(a)N$. If $q = r > 1$

$$\sum_{n \sim N} c_q(n) c_q(n-a) = \sum_{n \sim N} \sum_{j \le q}{}^{*} e_q(jn) \sum_{s \le q}{}^{*} e_q(-s(n-a)),$$

using the definition of Ramanujan sum (and $c_q(-m) = c_q(m)$ here). Exchanging the sums, it's

$$\sum_{j \le q}{}^{*} \sum_{s \le q}{}^{*} e_q(as) \sum_{n \sim N} e_q((j-s)n) = N c_q(a) + \mathcal{O}\left(\sum_{j \neq s} \sum \frac{1}{\left\| \frac{j-s}{q} \right\|} \right),$$

using the geometric sum estimate (e.g. [D,ch.25]); the thesis, since the errors (with $j, s (\bmod q)$) are

$$\mathcal{O}\left(\sum_{j} \sum_{0 < |\Delta| < q} \frac{1}{\|\frac{\Delta}{q}\|} \right) = \mathcal{O}\left(q \left(\sum_{0 < \Delta \le q/2} \frac{q}{\Delta} + \sum_{q/2 < \Delta < q} \frac{1}{\left| \|(\Delta - \frac{q}{2})\frac{1}{q}\| - \frac{1}{2} \right|} \right) \right) = \mathcal{O}\left(q^2 \log(q+1) \right).$$

When $q \neq r$ we apply the (quoted) formula $c_q(n) = \sum_{d|q, d|n} d\mu(q/d)$ in order to write

$$\sum_{n \sim N} c_q(n) c_r(n-a) = \sum_{g|q} g\mu(\frac{q}{g}) \sum_{s|r} s\mu(\frac{r}{s}) \sum_{\substack{n \sim N \\ n \equiv 0 (\bmod\ g) \\ n \equiv a (\bmod\ s)}} 1 = \sum_{\ell | a} \sum_{g|q} g\mu(\frac{q}{g}) \sum_{\substack{s|r \\ (s,g)=\ell}} s\mu(\frac{r}{s}) \sum_{\substack{n \sim N \\ n \equiv 0 (\bmod\ g) \\ n \equiv a (\bmod\ s)}} 1$$

$$= \sum_{\ell | a} \ell^2 \sum_{g^* \ell | q} g^* \mu(\frac{q}{g^* \ell}) \sum_{\substack{s^* \ell | r \\ (s^*, g^*)=1}} s^* \mu(\frac{r}{s^* \ell}) \sum_{\substack{n \sim N \\ n \equiv 0 (\bmod\ g^* \ell) \\ n \equiv a (\bmod\ s^* \ell)}} 1$$

$$= \sum_{\substack{\ell | a \\ \ell | q \\ \ell | r}} \ell^2 \sum_{g | \frac{q}{\ell}} g\mu(\frac{q}{g\ell}) \sum_{\substack{s | \frac{r}{\ell} \\ (s,g)=1}} s\mu(\frac{r}{s\ell}) \left(\frac{N}{\ell g s} + \mathcal{O}(1) \right) = N \sum_{\substack{\ell | a \\ \ell | q \\ \ell | r}} \ell \sum_{g | \frac{q}{\ell}} \mu(\frac{q}{g\ell}) \sum_{\substack{s | \frac{r}{\ell} \\ (s,g)=1}} \mu(\frac{r}{s\ell}) + \mathcal{O}(\sigma(q)\sigma(r))$$

$$= N \sum_{\substack{\ell | a \\ \ell | (q,r)}} \ell \sum_{g | \frac{q}{\ell}} \mu(\frac{q}{g\ell}) \sum_{\substack{s | \frac{r}{\ell} \\ (s,g)=1}} \mu(\frac{r}{s\ell}) + \mathcal{O}(qr \log(q+1)\log(r+1)),$$

where we used $\sigma(n) \overset{def}{=} \sum_{d|n} d = \sum_{d|n} \frac{n}{d} \ll n \sum_{d \le n} \frac{1}{d} \ll n \log(n+1)$. Also, exchanging sums,

$$\sum_{n \sim N} c_q(n) c_r(n-a) = N \sum_{\substack{\ell | a \\ \ell | (q,r)}} \ell \sum_{s | \frac{r}{\ell}} \mu(\frac{r}{s\ell}) \sum_{\substack{g | \frac{q}{\ell} \\ (g,s)=1}} \mu(\frac{q}{g\ell}) + \mathcal{O}(qr \log(q+1)\log(r+1)).$$

The inner-most sums are, "flipping" divisors, respectively, $\mu(\frac{r}{\ell})$ if $\frac{r}{\ell} | g$ (0 o.w.) & $\mu(\frac{q}{\ell})$ if $\frac{q}{\ell} | s$ (0 o.w.) and the same "main term" here "vanishes" (the following conditions in LHS & RHS imply $q = r$, absurd):

$$N \sum_{\substack{\ell | a \\ \ell | (q,r)}} \ell \sum_{\substack{g | \frac{q}{\ell} \\ \frac{r}{\ell} | g}} \mu(\frac{q}{g\ell}) \mu(\frac{r}{\ell}) = N \sum_{\substack{\ell | a \\ \ell | (q,r)}} \ell \sum_{\substack{s | \frac{r}{\ell} \\ \frac{q}{\ell} | s}} \mu(\frac{r}{s\ell}) \mu(\frac{q}{\ell}) = 0. \ \square$$

3. Proof of the results.

First of all, let's prove Theorem 1 (as Corollary 1 follows immediately from it).

We split the Ramanujan expansion of $f = g * \mathbf{1}$ (which is absolutely convergent, see Lemma 0):

$$S_f^{\pm}(x) = \sum_{q=1}^{\infty} a_q(f) \sum_{|n-x|\le h} c_q(n)\mathrm{sgn}(n-x) = S_1(x) + S_2(x),$$

say, where

$$S_1(x) \stackrel{def}{=} \sum_{q\le Q} a_q(f) \sum_{|n-x|\le h} c_q(n)\mathrm{sgn}(n-x) \ll L^2 \max_{D\ll Q} |S_D(x)|\,,$$

for $S_D(x) \stackrel{def}{=} \sum_{q\sim D} a_q(f) \sum_{|n-x|\le h} c_q(n)\mathrm{sgn}(n-x)$ and

$$S_2(x) \stackrel{def}{=} \sum_{q>Q} a_q(f) \sum_{|n-x|\le h} c_q(n)\mathrm{sgn}(n-x).$$

The mean-square of these tails are estimated using $K = (1+1/\varepsilon)^2$ (from Lemma 0) into Lemma 1:

$$\sum_{x\sim N} |S_2(x)|^2 \ll \left(1+\frac{1}{\varepsilon}\right)^4 Nh^2 \frac{L^3}{Q^{2\varepsilon}}.$$

In the same way, applying Lemma 2 we get:

$$\sum_{x\sim N} |S_1(x)|^2 \ll \left(1+\frac{1}{\varepsilon}\right)^2 hL^2 \max_{D\ll Q} \left(\frac{N}{D^{2\varepsilon}} + D^{2-2\varepsilon}\right) \ll_{\varepsilon} NhL^2 + Q^{2-2\varepsilon}hL^2.$$

For the sake of clarity we do not seek optimal bounds. Hence, choosing $Q = \sqrt{NhL}$ we obtain

$$\sum_{x\sim N} \left|S_f^{\pm}(x)\right|^2 \ll \left(1+\frac{1}{\varepsilon}\right)^4 Nh^2 \left(\frac{L^2}{h} + \frac{L^{3-\varepsilon}}{N^{\varepsilon}h^{\varepsilon}}\right). \ \square$$

Now, let's prove Theorem 2 (again, its Corollary is straightforward). For the same reason, even here our bounds are non-optimal. (And the implied constant will be the same as before.)

We perform a similar decomposition, but splitting at $Q = h$, this time (at last, use Cauchy inequality):

$$S_f^{\pm}(x) = \sum_{q=1}^{\infty} a_q(f) \sum_{|n-x|\le h} c_q(n)\mathrm{sgn}(n-x) \stackrel{def}{=} S_1(x) + S_2(x),$$

say, where (as before for K and this time applying Lemma 3 and Lemma 4)

$$\sum_{x\sim N} |S_1(x)|^2 = \sum_{q\,,\,r\le h}\sum a_q(f)\overline{a_r(f)} \sum_k W(k) \sum_{n\sim N} c_q(n)c_r(n-k) + \mathcal{O}_{\varepsilon}\left((Nh+h^3)\left(\sum_{q\le h}\frac{d(q)}{q^{1+\varepsilon}}\right)^2\right)$$

$$= N\mathfrak{S}_f(h) + \mathcal{O}_{\varepsilon}\left(h^2L^2\left(\sum_{q\le h}\frac{1}{q^{\varepsilon}}\right)^2 + (Nh+h^3)L^2\right) = N\mathfrak{S}_f(h) + \mathcal{O}_{\varepsilon}\left(h^4L^2 + (Nh+h^3)L^2\right),$$

from the bound [H-W] $\sum_{n\le x} d(n) \ll x\log x$ in partial summation; again, using Lemma 1

$$\sum_{x\sim N} |S_2(x)|^2 \ll_{\varepsilon} Nh^2\frac{L^3}{h^{2\varepsilon}} \Rightarrow I_f(N,h) = N\mathfrak{S}_f(h) + \mathcal{O}_{\varepsilon}\left(Nh^2\frac{L^{3/2}}{h^{\varepsilon}}\left(\frac{L}{\sqrt{h}} + \frac{hL}{\sqrt{N}} + \frac{L^{3/2}}{h^{\varepsilon}}\right)\right). \ \square$$

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