

ON THE CORRELATIONS, SELBERG INTEGRAL AND SYMMETRY OF SIEVE FUNCTIONS IN SHORT INTERVALS, III

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Abstract. We pursue the study of the arithmetic (real) function $f = g * \mathbf{1}$, with g “essentially bounded” and supported over the integers of $[1, Q]$. Applying (highly) non-trivial results [DFI] on bilinear forms of Kloosterman fractions, we obtain asymptotics for f “(auto-)correlations”, (together with f_1, f_2 “mixed” correlations), beyond the “classical” level of distribution (namely, the one obtainable applying Bombieri-Vinogradov level of distribution, in the arithmetic progressions, of $\lambda = 1/2$), up to level of distribution, say, $\lambda := \log Q / \log N < 1/2 + \delta$ (where the “gain” is $\delta = \frac{1}{190}$, here); we remark that such a level is unreachable, through the present level of distribution in the arithmetic progressions (i.e., the one quoted, due to Bombieri-Vinogradov Theorems!); as an application, we apply these asymptotics to the “Selberg integral” and the “symmetry integral” (and, also, their “mixed” counterparts) of our arithmetic f , in almost all short intervals $[x-h, x+h]$, $N \leq x \leq 2N$.

1. Introduction and statement of the results.

We continue the study of “SIEVE FUNCTIONS”, i.e. real arithmetic functions $f = g * \mathbf{1}$ (see hypotheses on $g \in \mathbb{R}$ in the sequel), in almost all the short intervals $[x-h, x+h]$ (i.e., almost all stands $\forall x \in [N, 2N]$, except $o(N)$ of them and short means, say, $h \rightarrow \infty$ and $h = o(N)$, as $N \rightarrow \infty$), started in [C1]. (Here, as usual, $\mathbf{1}(n) = 1$ is the constant-1 arithmetic function and $*$ is the Dirichlet product, esp., [T]). In order to study the sum of f values in a.a. (abbreviates almost all, now on) the intervals $[x-h, x+h]$, we introduce (compare [C1], in analogy with the classical Selberg integral, see [C-S]) the “SELBERG INTEGRAL” of f as:

$$J_f(N, h) \stackrel{def}{=} \int_N^{2N} \left| \sum_{x < n \leq x+h} f(n) - M_f(h) \right|^2 dx,$$

where (from heuristics in accordance with the classical case) we expect the “MEAN-VALUE” in $[x, x+h]$ to be

$$M_f(h) \stackrel{def}{=} h \sum_d \frac{g(d)}{d}, \quad \text{FROM: } h \left(\frac{1}{x} \sum_{n \leq x} f(n) \right) = \frac{h}{x} \sum_d g(d) \left[\frac{x}{d} \right] = h \sum_d \frac{g(d)}{d} + \mathcal{O} \left(\frac{h}{x} \sum_{d \leq Q} |g(d)| \right),$$

(here and in the sequel there will be no confusion with the symbol $[]$ for intervals or for the INTEGER PART) with convergence under the hypothesis $g(d) = 0, \forall d > Q$ (also, $d \leq 2N + h$, here). Assuming Q smaller than x (in the sequel), we recover $M_f(h)$. Selberg integral “weights” the values of f in a.a. $[x, x+h]$.

We study, also, their symmetry (around x) in $[x-h, x+h]$, through the “SYMMETRY INTEGRAL” of f :

$$I_f(N, h) \stackrel{def}{=} \int_N^{2N} \left| \sum_{|n-x| \leq h} \text{sgn}(n-x) f(n) \right|^2 dx$$

(the “SIGN” function is, as usual, $\text{sgn}(0) \stackrel{def}{=} 0, \text{sgn}(r) \stackrel{def}{=} \frac{r}{|r|}, \forall r \neq 0$). Motivations for their study are in [C].

We’ll improve the results given in [C1] for these integrals, applying again their expression as weighted sums (with weights, resp. S for J_f and W for I_f , see [C1] Lemmas 1 and 2) of the f “(auto-)correlations”.

The CORRELATION OF f IS DEFINED AS ($\forall a \in \mathbb{Z}, a \neq 0, a = o(N)$ here)

$$\mathcal{C}_f(a) \stackrel{def}{=} \sum_{n \sim N} f(n) f(n-a) = \sum_{\ell|a} \sum_{(d,q)=1} \sum_{\ell d} g(\ell d) g(\ell q) \frac{1}{q} \left(\left[\frac{2N}{\ell d} \right] - \left[\frac{N}{\ell d} \right] \right) + R_f(a)$$

(hereon $(a, b) = 1$ means a, b are COPRIME and $x \sim X$ is $X < x \leq 2X$), WHERE $R_f(a)$ IS AN ERROR-TERM (defined, for example, in [C1] from the orthogonality of the additive characters [V]).

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(Hereon, as usual, $F = o(G) \stackrel{def}{\iff} \lim F/G = 0$ and $F = \mathcal{O}(G) \stackrel{def}{\iff} \exists c > 0 : |F| \leq cG$ are Landau's notation. Also, when c depends on ε , we'll write $F = \mathcal{O}_\varepsilon(G)$ or, like Vinogradov, $F \ll_\varepsilon G$). We call an arithmetical function ESSENTIALLY BOUNDED when, $\forall \varepsilon > 0$, its n -th value is at most $\mathcal{O}_\varepsilon(n^\varepsilon)$ and we'll write $\lll 1$; i.e.,

$$F(N) \lll G(N) \stackrel{def}{\iff} \forall \varepsilon > 0 \quad F(N) \ll_\varepsilon N^\varepsilon G(N) \quad (\text{as } N \rightarrow \infty)$$

e.g., the divisor function $d(n)$ is essentially bounded (like all divisor functions d_k and many arithmetic functions); we remark : $f = g * \mathbf{1}$ is essentially bounded if and only if g is (from Möbius inversion, see [D]).

We need still some definitions, to express in a more abbreviate form the magnitude of our variables, from the point of view of exponents.

Our "MAIN VARIABLE" is $N \rightarrow \infty$.

We say that we're working in intervals of WIDTH $\theta \in [0, 1]$ WHEN $h = [N^\theta]$; actually, we'll follow the CONVENTION that any INEQUALITY on θ is SHARP, meaning that, for example, $\theta < 1$ means $\exists \delta > 0$ (absolute constant, no dependence on other variables!) such that $\theta < 1 - \delta$ (in fact, we will work in short intervals).

We call an arithmetic function f of LEVEL $\lambda \in [0, 1]$ IF $f = g * \mathbf{1}$ (i.e., $g = f * \mu$), with support of g contained into $[1, Q]$ and $Q = [N^\lambda]$; actually, we'll follow the SAME CONVENTION FOR INEQUALITIES on λ : esp., $\lambda < 1$ means that there's an absolute $\delta > 0$ such that $\lambda < 1 - \delta$ (this will also be assumed henceforth). (These two conventions will avoid introducing unuseful absolute constants.)

The terminology "LEVEL" comes from "LEVEL OF DISTRIBUTION IN THE ARITHMETIC PROGRESSIONS", that we'll abbreviate AP-level, where writing now $\lambda_{AP}(f)$ for this latter level of the function f means that

$$(\lambda_{AP}(f)) \quad \sum_{q \leq Q} \max_{(a,q)=1} \left| \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} f(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \leq x \\ (n,q)=1}} f(n) \right| \ll_\varepsilon x^{1-\varepsilon}$$

holds for $Q \leq x^{\lambda_{AP}(f)-\delta}$ ($\delta > 0$ small, depending on $\varepsilon > 0$). For example, Bombieri-Vinogradov Theorem gives $\lambda_{AP}(\Lambda) = \frac{1}{2}$, for the von-Mangoldt function $\Lambda(n) \stackrel{def}{=} \log p, \forall n = p^k$ ($k \in \mathbb{N}, p$ prime), $\stackrel{def}{=} 0$ otherwise.

Classical arguments (see the AMS Memoirs 538, i.e. Elliott's Monograph on Correlations) give, from the level of distribution $\lambda_{AP}(f)$ above for f , an asymptotic formula for the (auto-)correlations of the same f , whenever it's of the kind

$$f(n) = \sum_{\substack{q|n \\ q \leq Q}} g(q), \quad Q \ll N^{\lambda_{AP}(f)-\delta}$$

I.E., FOR THE SAME FUNCTION, AP-LEVEL λ GIVES LEVEL λ FOR THE CORRELATIONS.

Unfortunately, there seems to be no way (not only because not yet in the literature!) to go the other way round: i.e., the information of having (for the same f) AP-level λ seems to be much stronger than knowing f has level (for correlations) λ . For example, if this would be possible, our Main Lemma for the correlations would give a bound like Bombieri-Vinogradov's, but for level $\lambda > 1/2$ ($= \frac{1}{2} + \frac{1}{190}$: see next section).

However, our Main Lemma (next section) and the consequences for Selberg & symmetry integrals, i.e. our Main Theorem & our Corollary, give a substitute for the (up to now, unknown) AP-level over $1/2$ (even if only slightly above, i.e. $1/190$ over). This is for the autocorrelations, but the Lemma is even more general.

We give our "main" result. (Here we explicitly isolate $\Delta > 0$ to highlight the gain w.r.t. $\Delta = 0$ in [C1])

MAIN THEOREM. Fix $\delta > 0$ SMALL. Let $N, h, Q \in \mathbb{N}$, be such that $Q, h \rightarrow \infty$ and WIDTH $0 < \theta < 1/2$, as $N \rightarrow \infty$. Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be ESSENTIALLY BOUNDED, with LEVEL $\frac{1}{2} < \lambda < 1$. Set $\Delta = \frac{1}{48}$. Then

$$J_f(N, h) \lll Nh + N^\delta Q^{2-\Delta} h^2 + N^{1-2\delta/3} h^2 + Qh^2, \quad I_f(N, h) \lll Nh + N^\delta Q^{2-\Delta} h^2 + N^{1-2\delta/3} h^2.$$

Remark. We explicitly point out that our Theorem is proved, here, for levels $\lambda > 1/2$, since [C1] implies non-trivial estimates $J_f(N, h) \ll \frac{Nh^2}{N^\varepsilon}$ and $I_f(N, h) \ll \frac{Nh^2}{N^\varepsilon}$ for both integrals, with level up to $\lambda < \frac{1+\theta}{2}$ ($\theta > 0$ is the WIDTH); hence, we're actually using our previous Corollary in order to get the present one.

In fact, an immediate consequence of our Theorem and of the Corollary of [C1] is the following

COROLLARY. Fix $0 < \theta < 1/2$, $0 < \lambda < \max(\frac{1+\theta}{2}, \frac{1}{2} + \frac{1}{190})$. Let $N, h, Q \in \mathbb{N}$ be such that $h = [N^\theta]$, $Q = [N^\lambda]$. Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be ESSENTIALLY BOUNDED, with $f = g * \mathbf{1}$ and $g(q) = 0 \forall q > Q$. Then $\exists \varepsilon_0 = \varepsilon_0(\theta, \lambda) > 0$ (depending only on θ, λ) such that

$$J_f(N, h) \ll_{\varepsilon_0} N h^2 N^{-\varepsilon_0}, \quad I_f(N, h) \ll_{\varepsilon_0} N h^2 N^{-\varepsilon_0}.$$

Here the improvement w.r.t. [C1] level $\frac{1+\theta}{2}$ is present only when $0 < \theta < \frac{1}{95}$ (i.e., “very short” intervals).

Now we come to the corresponding results for “mixed Selberg integrals” and for “mixed symmetry integrals” (see below), obtained from asymptotic results (see main Lemma at next section) for the following “MIXED CORRELATIONS” (again, same hypotheses on a):

$$\mathcal{C}_{f_1, f_2}(a) \stackrel{\text{def}}{=} \sum_{n \sim N} f_1(n) f_2(n - a).$$

In fact, using the same procedure as for the “pure” quantities (integrals & correlations) above, we may express as weighted sums of mixed correlations the MIXED SELBERG INTEGRAL (use the “Selberg weight” S , see [C1])

$$J_{f_1, f_2}(N, h) \stackrel{\text{def}}{=} \int_N^{2N} \left(\sum_{x < n \leq x+h} f_1(n) - M_{f_1}(h) \right) \left(\sum_{x < m \leq x+h} f_2(m) - M_{f_2}(h) \right) dx,$$

where (see the above), setting $f_1 = g_1 * \mathbf{1}$ and $f_2 = g_2 * \mathbf{1}$, we expect the “MEAN-VALUES” in $[x, x+h]$ to be

$$M_{f_1}(h) \stackrel{\text{def}}{=} h \sum_d \frac{g_1(d)}{d}, \quad M_{f_2}(h) \stackrel{\text{def}}{=} h \sum_d \frac{g_2(d)}{d};$$

and the MIXED SYMMETRY INTEGRAL (use the “symmetry weight” W , see [C1])

$$I_{f_1, f_2}(N, h) \stackrel{\text{def}}{=} \int_N^{2N} \left(\sum_{|n-x| \leq h} \text{sgn}(n-x) f_1(n) \right) \left(\sum_{|m-x| \leq h} \text{sgn}(m-x) f_2(m) \right) dx.$$

We give our “auxiliary” results. (A motivation for studying “mixed” counterparts is in [C3], esp.)

PROPOSITION 1. Fix WIDTH $0 < \theta < 1/2$, LEVELS $\frac{1}{2} < \lambda_1 \leq \lambda_2 < 1$. Let $N, h, D, Q \in \mathbb{N}$ be such that $h = [N^\theta]$, $D = [N^{\lambda_1}]$, $Q = [N^{\lambda_2}]$ as $N \rightarrow \infty$. Let $f_1, f_2 : \mathbb{N} \rightarrow \mathbb{R}$ be ESSENTIALLY BOUNDED, with $f_1 = g_1 * \mathbf{1}$, $f_2 = g_2 * \mathbf{1}$, $g_1(d) = 0 \forall d > D$, $g_2(q) = 0 \forall q > Q$. Then

$$J_{f_1, f_2}(N, h) \ll N h + N^\delta (DQ)^{\frac{7}{8}} Q^{\frac{11}{48}} h^2 + N^{1-\frac{2\delta}{3}} h^2 + Q h^2, \quad I_{f_1, f_2}(N, h) \ll N h + N^\delta (DQ)^{\frac{7}{8}} Q^{\frac{11}{48}} h^2 + N^{1-\frac{2\delta}{3}} h^2.$$

Remark. We explicitly point out that our Proposition 1 is proved, here, for levels $\lambda > 1/2$, since [C1] Corollary extension to mixed counterparts implies the present form of this Corollary.

In fact, an immediate consequence of (a “mixed analogue” of [C1] Corollary and of) Proposition 1 is

PROPOSITION 2. Fix WIDTH $0 < \theta < 1/2$, LEVELS $\lambda_2 \geq \lambda_1 > 0$ SATISFYING $\lambda_1 + \lambda_2 < 1$ OR $42\lambda_1 + 53\lambda_2 < 48$. Let $N, h, D, Q \in \mathbb{N}$ be such that $h = [N^\theta]$, $D = [N^{\lambda_1}]$, $Q = [N^{\lambda_2}]$. Let $f_1, f_2 : \mathbb{N} \rightarrow \mathbb{R}$ be ESSENTIALLY BOUNDED, with $f_1 = g_1 * \mathbf{1}$, $f_2 = g_2 * \mathbf{1}$ and $g_1(d) = 0 \forall d > D$, $g_2(q) = 0 \forall q > Q$. Then $\exists \varepsilon_0 = \varepsilon_0(\theta, \lambda_1, \lambda_2) > 0$ (depending only on $\theta, \lambda_1, \lambda_2$) such that

$$J_{f_1, f_2}(N, h) \ll_{\varepsilon_0} N h^2 N^{-\varepsilon_0}, \quad I_{f_1, f_2}(N, h) \ll_{\varepsilon_0} N h^2 N^{-\varepsilon_0}.$$

The paper is organized as follows:

- ◊ we will give our Lemmas in the next section;
- ◊ we will show the Propositions in section 3;
- ◊ then, we’ll prove our Theorem in section 4.

2. Lemmas.

We will ABBREVIATE $n \equiv a(\text{mod } q)$ with $n \equiv a(q)$.

The following Lemma gives a strong level ($> 1/2$) for the autocorrelations, together with a fairly general asymptotic formula for (mixed-)correlations.

MAIN LEMMA. FIX $\delta > 0$. Let $N, h, D, Q \in \mathbb{N}$ be such that $D, Q \rightarrow \infty$ and we have WIDTH $0 < \theta < 1/2$, as $N \rightarrow \infty$. Let $g_1, g_2 : \mathbb{N} \rightarrow \mathbb{R}$ be essentially bounded, with supports (resp.) into $[1, D]$, $[1, Q]$, with $1 \leq D \leq Q \ll N^{1-\delta}$. Let $f_1 = g_1 * \mathbf{1}$, $f_2 = g_2 * \mathbf{1}$. Then, UNIFORMLY $\forall a \in \mathbb{Z}$, $0 < |a| = o(N)$,

$$\forall \varepsilon > 0 \quad \frac{\mathcal{C}_{f_1, f_2}(a) + \mathcal{C}_{f_1, f_2}(-a)}{2} = N \sum_{\ell|a} \frac{1}{\ell} \sum_{(d, q)=1} \frac{g_1(\ell d)}{d} \frac{g_2(\ell q)}{q} + \mathcal{O}_\varepsilon \left(N^{\delta+\varepsilon} (DQ)^{\frac{7}{8}} Q^{\frac{11}{48}} + N^{1-2\delta/3+\varepsilon} \right).$$

In particular, choosing $f_1 = f_2 = f = g * \mathbf{1}$ we get LEVEL $\lambda = \frac{1}{2} + \frac{1}{190}$ FOR (AUTO-)CORRELATIONS, SINCE

$$\forall \varepsilon > 0 \quad \mathcal{C}_f(a) = N \sum_{\ell|a} \frac{1}{\ell} \sum_{(d, q)=1} \frac{g(\ell d)}{d} \frac{g(\ell q)}{q} + \mathcal{O}_\varepsilon \left(N^{\delta+\varepsilon} Q^{\frac{95}{48}} + N^{1-2\delta/3+\varepsilon} \right).$$

Remark. We explicitly point out that our Lemma improves, also in the mixed case, [C1] non-trivial bounds.

The Lemma is proved, through a (rather) straightforward application of the following Lemmas. First, some definitions.

Define $\bar{d}(\text{mod } q)$, the RECIPROCAL residue of $d(\text{mod } q)$, $\forall (d, q) = 1$, as $\bar{d}d \equiv 1(\text{mod } q)$ and, $\forall a \in \mathbb{Z}$, $a \neq 0$,

$$\begin{aligned} R_{D, Q}(|a|) & \stackrel{\text{def}}{=} R(|a|, g_1, g_2, D, Q, N) \\ & - \sum_{\substack{\ell|a \\ b:=\frac{|a|}{\ell}}} \sum_{d \sim \frac{D}{\ell}} g_1(\ell d) \sum_{\substack{q \sim \frac{Q}{\ell} \\ (q, d)=1}} g_2(\ell q) \left(B_1 \left(\frac{[2N/\ell d] - \bar{d}b}{q} \right) - B_1 \left(\frac{[N/\ell d] - \bar{d}b}{q} \right) \right) \\ & - \sum_{\substack{\ell|a \\ b:=\frac{|a|}{\ell}}} \sum_{d \sim \frac{D}{\ell}} g_1(\ell d) \sum_{\substack{q \sim \frac{Q}{\ell} \\ (q, d)=1}} g_2(\ell q) \left(B_1 \left(\frac{[2N/\ell d] + \bar{d}b}{q} \right) - B_1 \left(\frac{[N/\ell d] + \bar{d}b}{q} \right) \right), \end{aligned}$$

where we recall the definition of the FIRST BERNOULLI FUNCTION (1-periodicized of the 1st Bernoulli POLYNOMIAL):

$$B_1(\alpha) \stackrel{\text{def}}{=} \{\alpha\} - 1/2, \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Z}$$

(here $\{\alpha\} \stackrel{\text{def}}{=} \alpha - [\alpha]$ is the FRACTIONAL PART of $\alpha \in \mathbb{R}$) and $B_1 = 0$ on \mathbb{Z} . Hereon we use $e(\beta) \stackrel{\text{def}}{=} e^{2\pi i \beta}$ [V] with $e_q(n) \stackrel{\text{def}}{=} e(n/q) \forall n \in \mathbb{Z}$, $\forall q \in \mathbb{N}$, and

$$\|\alpha\| \stackrel{\text{def}}{=} \min_{n \in \mathbb{Z}} |\alpha - n|,$$

the DISTANCE FROM THE INTEGERS.

In what follows we'll avoid the cases $D, Q \ll |a|$ (assuming $\frac{D}{\ell}, \frac{Q}{\ell} \gg 1$ in following Lemma proof), since, esp.,

$$a > 0, D \ll a \text{ AND } g_1, g_2 \ll 1 \Rightarrow R_{D, Q}(|a|) \ll \sum_{\ell|a} \frac{aQ}{\ell^2} \ll aQ$$

can be considered a very good bound (recall we'll use it when $D, Q = o(N^{1-\delta}/|a|)$, giving $R_{D, Q}(|a|) \ll N^{1-\delta}$).

However, this restriction will be implicit (compare Lemma B [C2]) in our

LEMMA A. Fix $\delta > 0$ enough small (SAY $\delta < \frac{1}{100}$). Let $N \in \mathbb{N}$ and $a \in \mathbb{Z}$, $a \neq 0$, $|a| = o(N)$; assume $g_1, g_2 : \mathbb{N} \rightarrow \mathbb{R}$ supported into (resp.) $]D, 2D],]Q, 2Q]$, where $D, Q \in \mathbb{N}$ with $DQ \gg N^{1-2\delta/3}$ and $D, Q = o(N^{1-\delta})$ when $N \rightarrow \infty$. Also, assume that g_1, g_2 are ESSENTIALLY BOUNDED. Then $\forall \varepsilon > 0$

$$R_{D,Q}(|a|) = \frac{2}{\pi} \sum_{\substack{\ell|a \\ b:=\frac{|a|}{\ell}}} \sum_{d \sim \frac{D}{\ell}} g_1(\ell d) \sum_{\substack{q \sim \frac{Q}{\ell} \\ (q,d)=1}} g_2(\ell q) \sum_{j \leq J} \frac{\sin \frac{2\pi[2N/\ell d]j}{q} - \sin \frac{2\pi[N/\ell d]j}{q}}{j} \cos \frac{2\pi j \bar{d} b}{q} + \mathcal{O}_\varepsilon(N^{1-\delta+\varepsilon}),$$

where $J = J(\ell, d, q, \delta, N) \stackrel{def}{=} \lfloor \ell d q / N^{1-\delta} \rfloor$ (see that $J = o(Q/\ell)$, $\forall \ell|a$, from our assumptions).

PROOF. We only give a sketch: details may be found in [C2]. Assume $\ell \leq \frac{DQ}{LN^{1-\delta}}$, abbrev. $L := \log N$, since

$$\begin{aligned} & - \sum_{\substack{\ell|a \\ \ell > \frac{DQ}{LN^{1-\delta}}}} \sum_{d \sim \frac{D}{\ell}} g_1(\ell d) \sum_{\substack{q \sim \frac{Q}{\ell} \\ (q,d)=1}} g_2(\ell q) \left(B_1 \left(\frac{[2N/\ell d] - \bar{d} b}{q} \right) - B_1 \left(\frac{[N/\ell d] - \bar{d} b}{q} \right) \right) \\ & - \sum_{\substack{\ell|a \\ \ell > \frac{DQ}{LN^{1-\delta}}}} \sum_{d \sim \frac{D}{\ell}} g_1(\ell d) \sum_{\substack{q \sim \frac{Q}{\ell} \\ (q,d)=1}} g_2(\ell q) \left(B_1 \left(\frac{[2N/\ell d] + \bar{d} b}{q} \right) - B_1 \left(\frac{[N/\ell d] + \bar{d} b}{q} \right) \right) \end{aligned}$$

(with $b := |a|/\ell$ as above) may be bounded, trivially ($B_1 \ll 1$):

$$\lll \sum_{\substack{\ell|a \\ \ell > \frac{DQ}{LN^{1-\delta}}}} \frac{DQ}{\ell^2} \lll N^{1-\delta}.$$

We'll ignore this limitation in our final sum, (again) because of this bound, since the $j \leq J$ sum is $\ll L$. The FINITE FOURIER EXPANSION (see, esp., Lemma 3 in the file "example", on my dept.webpage):

$$B_1 \left(\frac{n}{q} \right) = -\frac{1}{q} \sum_{j \leq \frac{q}{2}} \cot \frac{\pi j}{q} \sin \frac{2\pi j n}{q} \quad \forall q \in \mathbb{N}, \quad \forall n \in \mathbb{Z}$$

(see that, whenever $q = 1$, the sum is empty and in fact $B_1 = 0$) gives (now on $b = \frac{|a|}{\ell}$ in the sums)

$$R_{D,Q}(|a|) = \mathcal{O}_\varepsilon(N^{1-\delta+\varepsilon}) + 2 \sum_{\substack{\ell|a, \ell \leq \frac{DQ}{LN^{1-\delta}}}} \sum_{d \sim \frac{D}{\ell}} g_1(\ell d) \sum_{\substack{q \sim \frac{Q}{\ell} \\ (q,d)=1}} \frac{g_2(\ell q)}{q} \sum_{j \leq \frac{q}{2}} F \left(\frac{j}{q} \right),$$

WITH $F \left(\frac{j}{q} \right) = F_{b\bar{d}, N/\ell d} \left(\frac{j}{q} \right)$ defined as

$$F_{b\bar{d}, N/\ell d} \left(\frac{j}{q} \right) \stackrel{def}{=} \cot \frac{\pi j}{q} \left(\sin \frac{2\pi[2N/\ell d]j}{q} - \sin \frac{2\pi[N/\ell d]j}{q} \right) \cos \frac{2\pi j \bar{d} b}{q}.$$

Then, SPLIT the sum over j at $J = \lfloor \ell d q / N^{1-\delta} \rfloor$ (see that, now on $\ell \leq \frac{DQ}{LN^{1-\delta}}$, which implies $J \rightarrow \infty$):

$$\sum_{j \leq \frac{q}{2}} F \left(\frac{j}{q} \right) = \sum_{j \leq J} F \left(\frac{j}{q} \right) + \sum_{J < j \leq \frac{q}{2}} F \left(\frac{j}{q} \right)$$

and

$$\sum_{j \leq J} F \left(\frac{j}{q} \right) = \sum_{j \leq J} \left(\frac{q}{\pi j} + \sum_{n=1}^{K-1} a_n \left(\frac{j}{q} \right)^n \right) \left(\sin \frac{2\pi[2N/\ell d]j}{q} - \sin \frac{2\pi[N/\ell d]j}{q} \right) \cos \frac{2\pi j \bar{d} b}{q} + \mathcal{O} \left(\frac{J^{K+1}}{q^K} \right)$$

$$= \frac{q}{\pi} \sum_{j \leq J} \frac{\sin \frac{2\pi[2N/\ell d]j}{q} - \sin \frac{2\pi[N/\ell d]j}{q}}{j} \cos \frac{2\pi j \bar{d} b}{q} + R(b, d, q, J, \ell, N) + \mathcal{O}\left(N^{-K\delta} \frac{Dq}{N^{1-\delta}}\right),$$

where the a_n are due to the ‘‘cot’’ function series expansion and $R(b, d, q, J, \ell, N)$ is a remainder term, that we’ll treat soon after the main term, here. (It is vital here, to get $\ll N^{1-\delta}$ at last, to let $K\delta > 1$.) Applying PARTIAL SUMMATION [T] to the bound $\sum_n e(n\alpha) \ll \frac{1}{\|\alpha\|}$ [D, ch.26] for the exponentials,

$$\begin{aligned} \sum_{J < j \leq \frac{q}{2}} F\left(\frac{j}{q}\right) &\ll q \int_J^{\frac{q}{2}} \left| \sum_{J < j \leq v} e_q(j([2N/\ell d] \pm \bar{d}b)) \right| + \left| \sum_{J < j \leq v} e_q(j([N/\ell d] \pm \bar{d}b)) \right| \frac{dv}{v^2} \ll \\ &\ll \frac{q}{J} \left(\frac{1}{\left\| \frac{[2N/\ell d] \pm \bar{d}b}{q} \right\|} + \frac{1}{\left\| \frac{[N/\ell d] \pm \bar{d}b}{q} \right\|} \right), \end{aligned}$$

whenever none of these four distances from the integers vanish, i.e. $q \nmid d \left[\frac{cN}{\ell d} \right] \pm b \ \forall c = 1, 2$, while

$$q \mid d \left[\frac{cN}{\ell d} \right] \pm b \Rightarrow \sum_{J < j \leq \frac{q}{2}} F\left(\frac{j}{q}\right) \ll q, \text{ FROM THE TRIVIAL } \sum_{J < j \leq v} e(j\alpha) \ll v, \ \forall v \in [J, q/2].$$

Thus, it remains to estimate, say, assuming implicitly $(d, q) = 1$ whenever \bar{d} appears,

$$\begin{aligned} \sum_{\substack{\ell \mid a \\ \ell \leq \frac{DQ}{LN^{1-\delta}}}} \sum_{q \sim \frac{Q}{\ell}} \frac{g_2(\ell q)}{q} \sum_{\substack{d \sim \frac{D}{\ell} \\ (d, q) = 1}} g_1(\ell d) \sum_{J < j \leq \frac{q}{2}} F\left(\frac{j}{q}\right) &\ll \frac{N^{1-\delta}}{Q^2} \sum_{\ell \mid a} \ell \sum_{\substack{d \sim \frac{D}{\ell}, q \sim \frac{Q}{\ell} \\ d \left[\frac{cN}{\ell d} \right] \neq \pm b(q)}} \frac{1}{\left\| \frac{[cN/\ell d] \pm \bar{d}b}{q} \right\|} + \sum_{\ell \mid a} \sum_{\substack{d, q \sim \frac{Q}{\ell} \\ q \mid d \left[\frac{cN}{\ell d} \right] \pm b}} 1 \\ &\ll \frac{N^{1-\delta}}{Q^2} \sum_{\ell \mid a} \ell \sum_{q \sim \frac{Q}{\ell}} \sum_{0 < |r| \leq \frac{q}{2}} \frac{q}{|r|} \sum_{\substack{d \sim \frac{D}{\ell} \\ d \left[\frac{cN}{\ell d} \right] \pm b = r d(q)}} 1 + \sum_{\ell \mid a} \left(\sum_{\substack{d \sim \frac{D}{\ell} \\ d \nmid b}} \sum_{q \mid d \left[\frac{cN}{\ell d} \right] \pm b} 1 + \sum_{d \mid b} \frac{Q}{\ell} \right) \ll \\ &\ll \frac{N^{1-\delta}}{Q^2} \sum_{\ell \mid a} \ell \left(\sum_{d \sim \frac{D}{\ell}, d \nmid b} \frac{Q}{\ell} + \sum_{d \mid b} \left(\frac{Q}{\ell} \right)^2 \right) + Q \ll N^{1-\delta}. \end{aligned}$$

Finally, use the same bounds for distances to \mathbb{Z} for $R(b, d, q, J, \ell, N)$, from partial summation $\forall n \leq K$:

$$\begin{aligned} \frac{1}{q^n} \sum_{j \leq J} j^n \left(\sin \frac{2\pi[2N/\ell d]j}{q} - \sin \frac{2\pi[N/\ell d]j}{q} \right) \cos \frac{2\pi j \bar{d} b}{q} &\ll \frac{J^n}{q^n} \max_{v \leq J} \left| \sum_{j \leq v} e_q(j([cN/\ell d] \pm \bar{d}b)) \right| \ll \\ &\ll \left(\frac{D}{N^{1-\delta}} \right)^n \left(\frac{1}{\left\| \frac{[2N/\ell d] \pm \bar{d}b}{q} \right\|} + \frac{1}{\left\| \frac{[N/\ell d] \pm \bar{d}b}{q} \right\|} \right). \quad \square \end{aligned}$$

The following is a simple application of Theorem 2 [DFI] on bilinear forms of Kloosterman fractions:

LEMMA B. *Let $N, D, Q \in \mathbb{N}$, $D \leq Q \leq N$ and $k \in \mathbb{Z}$, $k \neq 0$, with $k \ll DQ$, as $D, Q \rightarrow \infty$; ASSUME that $g_1, g_2 : \mathbb{N} \rightarrow \mathbb{R}$ are ESSENTIALLY BOUNDED. Then*

$$\sum_{d \sim D} g_1(d) \sum_{\substack{q \sim Q \\ (q, d) = 1}} g_2(q) e_q(k\bar{d}) \ll (DQ)^{\frac{7}{8}} Q^{\frac{11}{18}}.$$

However, Lemma A and Lemma B together give

LEMMA C. Fix $\delta > 0$ enough small (SAY $\delta < \frac{1}{100}$). Let $N \in \mathbb{N}$ and $a \in \mathbb{Z}$, $a \neq 0$, $a \ll N^{1-\delta}$; assume $g_1, g_2 : \mathbb{N} \rightarrow \mathbb{R}$ supported into (resp.) $]D, 2D],]Q, 2Q]$, where $D, Q \in \mathbb{N}$ with $DQ \gg N^{1-2\delta/3}$ and $D, Q = o(N^{1-\delta})$ when $N \rightarrow \infty$. Also, assume that g_1, g_2 are ESSENTIALLY BOUNDED. Then

$$R_{D,Q}(|a|) \ll N^\delta (DQ)^{\frac{7}{8}} Q^{\frac{11}{48}} + N^{1-\delta}.$$

PROOF. We may confine to prove, from Lemma A (see its proof, too), $\forall c = 1, 2$, the following :

$$\Sigma(c) := \sum_{\substack{\ell|a, b := \frac{|a|}{\ell} \\ \ell \leq \frac{DQ}{LN^{1-\delta}}}} \sum_{d \sim \frac{D}{\ell}} g_1(\ell d) \sum_{\substack{q \sim \frac{Q}{\ell} \\ (q, d)=1}} g_2(\ell q) \sum_{j \leq \frac{4QD}{\ell N^{1-\delta}}} \frac{\sin \frac{2\pi[cN/\ell d]j}{q}}{j} \cos \frac{2\pi j \bar{d}b}{q} \ll N^\delta (DQ)^{\frac{7}{8}} Q^{\frac{11}{48}} + N^{1-\delta}.$$

In fact, recalling J definition (Lemma A),

$$\Sigma(c) = \sum_{\substack{\ell|a, b := \frac{|a|}{\ell} \\ \ell \leq \frac{DQ}{LN^{1-\delta}}}} \sum_{d \sim \frac{D}{\ell}} g_1(\ell d) \sum_{\substack{q \sim \frac{Q}{\ell} \\ (q, d)=1}} g_2(\ell q) \left(\sum_{j \leq \frac{4QD}{\ell N^{1-\delta}}} \frac{\sin \frac{2\pi c N j}{\ell d q}}{j} \cos \frac{2\pi j \bar{d}b}{q} + \mathcal{O} \left(\frac{1}{J \left\| \frac{[cN/(\ell d)] \pm \bar{d}b}{q} \right\|} \right) \right),$$

again from partial summation & previous arguments, whence the remainder is

$$\ll \sum_{\ell|a, b := \frac{|a|}{\ell}} \sum_{d \sim \frac{D}{\ell}} \sum_{\substack{q \sim \frac{Q}{\ell} \\ (q, d)=1}} \frac{N^{1-\delta}}{\ell d q} \frac{1}{\left\| \frac{[cN/(\ell d)] \pm \bar{d}b}{q} \right\|} \ll N^{1-\delta}.$$

From: $\sin \frac{2\pi[cN/\ell d]j}{q} = \cos \frac{2\pi\{cN/\ell d\}j}{q} \sin \frac{2\pi c N j}{\ell d q} - \sin \frac{2\pi\{cN/\ell d\}j}{q} \cos \frac{2\pi c N j}{\ell d q}$, $\forall c = 1, 2$,

$$\Sigma(c) = \Sigma_0(c) - \Sigma_1(c) - \Sigma_2(c),$$

SAY, where

$$\Sigma_0(c) := \sum_{\substack{\ell|a, b := \frac{|a|}{\ell} \\ \ell \leq \frac{DQ}{LN^{1-\delta}}}} \sum_{j \leq \frac{4QD}{\ell N^{1-\delta}}} \frac{1}{j} \sum_{d \sim \frac{D}{\ell}} g_1(\ell d) \sum_{\substack{q \sim \frac{Q}{\ell} \\ (q, d)=1}} g_2(\ell q) \sin \frac{2\pi c N j}{\ell d q} \cos \frac{2\pi j \bar{d}b}{q},$$

$$\Sigma_1(c) := \sum_{\substack{\ell|a, b := \frac{|a|}{\ell} \\ \ell \leq \frac{DQ}{LN^{1-\delta}}}} \sum_{d \sim \frac{D}{\ell}} g_1(\ell d) \sum_{\substack{q \sim \frac{Q}{\ell} \\ (q, d)=1}} g_2(\ell q) \sum_{j \leq \frac{4QD}{\ell N^{1-\delta}}} \frac{1 - \cos \frac{2\pi\{cN/\ell d\}j}{q}}{j} \sin \frac{2\pi c N j}{\ell d q} \cos \frac{2\pi j \bar{d}b}{q},$$

$$\Sigma_2(c) := \sum_{\substack{\ell|a, b := \frac{|a|}{\ell} \\ \ell \leq \frac{DQ}{LN^{1-\delta}}}} \sum_{d \sim \frac{D}{\ell}} g_1(\ell d) \sum_{\substack{q \sim \frac{Q}{\ell} \\ (q, d)=1}} g_2(\ell q) \sum_{j \leq \frac{4QD}{\ell N^{1-\delta}}} \frac{\sin \frac{2\pi\{cN/\ell d\}j}{q}}{j} \cos \frac{2\pi c N j}{\ell d q} \cos \frac{2\pi j \bar{d}b}{q}.$$

Then, partial summation [T], together with

$$\sum_{j \leq t} e \left(\frac{cNj}{\ell d q} \pm \frac{j \bar{d}b}{q} \right) \ll \frac{1}{\left\| \frac{cN/(\ell d) \pm \bar{d}b}{q} \right\|}$$

and Lemma A (see its proof) arguments to treat these distances to \mathbb{Z} give ($\forall c = 1, 2$)

$$|\Sigma_1(c)| + |\Sigma_2(c)| \ll \frac{D}{Q} + D \ll N^{1-\delta}.$$

Thus, we are left with $\Sigma_0(c)$ to bound. Apply partial summation over d :

$$\begin{aligned} \sum_{d \sim \frac{D}{\ell}} g_1(\ell d) \sum_{\substack{q \sim \frac{Q}{\ell} \\ (q,d)=1}} g_2(\ell q) \sin \frac{2\pi c N j}{\ell d q} \cos \frac{2\pi j \bar{d} b}{q} &= \sum_{d \sim \frac{D}{\ell}} g_1(\ell d) \sum_{\substack{q \sim \frac{Q}{\ell} \\ (q,d)=1}} g_2(\ell q) \sin \frac{\pi c N j}{D q} \cos \frac{2\pi j \bar{d} b}{q} + \\ &+ \frac{2\pi c N j}{Q} \int_{D/\ell}^{2D/\ell} \sum_{\frac{D}{\ell} < d \leq v} g_1(\ell d) \sum_{\substack{q \sim \frac{Q}{\ell} \\ (q,d)=1}} g_2(\ell q) \left(\frac{Q}{\ell q} \cos \frac{2\pi c N j}{\ell v q} \right) \cos \frac{2\pi j \bar{d} b}{q} \frac{dv}{v^2} \end{aligned}$$

whence, from Lemma B,

$$\Sigma_0(c) \ll \sum_{\substack{\ell | a \\ b := \frac{|a|}{\ell}}} \sum_{j \leq \frac{4QD}{\ell N^{1-\delta}}} \frac{1}{j} \left(1 + \frac{Nj\ell}{DQ} \right) \left(\frac{(DQ)^{\frac{7}{8}} Q^{\frac{11}{48}}}{\ell^{\frac{95}{48}}} \right) \ll N^\delta (DQ)^{\frac{7}{8}} Q^{\frac{11}{48}}. \quad \square$$

PROOF OF THE MAIN LEMMA. We start with the case $f_1 = f_2$, as our ‘‘INCIPIIT’’ resembles [C2] Thm.1 proof:

$$a > 0 \Rightarrow \mathcal{C}_{f_1, f_2}(-a) = \sum_{n \sim N} f_1(n) f_2(n+a) = \sum_{N+a < n \leq 2N+a} f_2(n) f_1(n-a) = \mathcal{C}_{f_2, f_1}(a) + \mathcal{O}_\varepsilon(N^\varepsilon a),$$

($\forall f_1, f_2$ ESSENTIALLY BOUNDED) GIVING (with a negligible contribute to our remainder)

$$a \neq 0, f_1 = f_2 \Rightarrow \frac{\mathcal{C}_{f_1, f_2}(a) + \mathcal{C}_{f_1, f_2}(-a)}{2} = \mathcal{C}_f(a) + \mathcal{O}_\varepsilon(N^\varepsilon |a|).$$

We can pass to the ‘‘mixed’’ case. We work in dyadic intervals: a dyadic argument applies, with all the logarithms into \ll , here. Also, assume $DQ \gg N^{1-\frac{2\delta}{3}}$, otherwise trivially $R_{D,Q}(|a|) \ll DQ \ll N^{1-\frac{2\delta}{3}}$. We open (i.e., $f_1 = g_1 * \mathbf{1}$ & $f_2 = g_2 * \mathbf{1}$) the mixed correlations, organizing with g.c.d.s, SAY $\ell := (d, q)$,

$$\begin{aligned} \frac{\mathcal{C}_{f_1, f_2}(a) + \mathcal{C}_{f_1, f_2}(-a)}{2} &= \frac{1}{2} \sum_{\ell | a} \sum_{q \sim Q} g_2(q) \sum_{\substack{d \sim D \\ (d,q)=\ell}} g_1(d) \left(\sum_{\substack{n \sim N \\ n \equiv 0(d), n \equiv a(q)}} 1 + \sum_{\substack{n \sim N \\ n \equiv 0(d), n \equiv -a(q)}} 1 \right) = \\ &= \sum_{\substack{\ell | a \\ b := \frac{|a|}{\ell}}} \sum_{d \sim \frac{D}{\ell}} g_1(\ell d) \sum_{\substack{q \sim \frac{Q}{\ell} \\ (q,d)=1}} g_2(\ell q) \frac{1}{2} \left(\sum_{\substack{m \sim \frac{N}{\ell d} \\ m \equiv db(q)}} 1 + \sum_{\substack{m \sim \frac{N}{\ell d} \\ m \equiv -db(q)}} 1 \right) = N \sum_{\ell | a} \frac{1}{\ell} \sum_{(d,q)=1} \sum_{\ell} \frac{g_1(\ell d)}{d} \frac{g_2(\ell q)}{q} + \frac{R_{D,Q}(|a|)}{2} \\ &\quad + \mathcal{O}_\varepsilon \left(N^\varepsilon \sum_{\substack{\ell | a \\ b := \frac{|a|}{\ell}}} \sum_{d \sim \frac{D}{\ell}} \sum_{\substack{q \sim \frac{Q}{\ell} \\ q | d \left[\frac{cN}{\ell d} \right] \pm b}} 1 \right), \end{aligned}$$

with (from $h = o(N) \Rightarrow d[cN/(\ell d)] \pm b \neq 0$, compare Lemma A proof) this last term $\ll D \ll N^{1-\delta}$, where, from Lemma C,

$$R_{D,Q}(|a|) \ll N^\delta (DQ)^{\frac{7}{8}} Q^{\frac{11}{48}} + N^{1-\delta}. \quad \square$$

3. Proof of the Propositions.

PROOF OF PROPOSITION 1. In order to express the mixed symmetry integral I_{f_1, f_2} as a sum of correlations,

$$\begin{aligned}
I_{f_1, f_2}(N, h) &= \int_N^{2N} \sum_{|n-x| \leq h} f_1(n) \operatorname{sgn}(n-x) \sum_{|m-x| \leq h} f_2(m) \operatorname{sgn}(m-x) dx = \\
&= \sum_{N-h < n, m \leq 2N+h} f_1(n) f_2(m) \int_{\substack{N \leq x \leq 2N \\ |x-n| \leq h \\ |x-m| \leq h}} \operatorname{sgn}(x-n) \operatorname{sgn}(x-m) dx = \\
&= \sum_{N+h < n, m \leq 2N-h} f_1(n) f_2(m) \int_{\substack{|t| \leq h \\ |t+(n-m)| \leq h}} \operatorname{sgn}(t) \operatorname{sgn}(t+(n-m)) dt + \mathcal{O}_\varepsilon(N^\varepsilon h^3) = \\
&= \sum_{N < n \leq 2N} f_1(n) \sum_{0 \leq |a| \leq 2h} f_2(n-a) \int_{\substack{|t| \leq h \\ |t+a| \leq h}} \operatorname{sgn}(t) \operatorname{sgn}(t+a) dt + \mathcal{O}_\varepsilon(N^\varepsilon h^3),
\end{aligned}$$

whence ignoring (write \sim for this) $\lll h^3$ terms (i.e., say, TAILS), we get (recall [C1] definition of W)

$$I_{f_1, f_2}(N, h) \sim \sum_{N < n \leq 2N} f_1(n) \sum_{0 \leq |a| \leq 2h} f_2(n-a) \int_{\substack{|t| \leq h \\ |t-a| \leq h}} \operatorname{sgn}(t) \operatorname{sgn}(t-a) dt = \sum_{0 \leq |a| \leq 2h} W(a) \sum_{n \sim N} f_1(n) f_2(n-a).$$

We have proved that, provided f_1, f_2 are REAL AND ESSENTIALLY BOUNDED (compare Lemma 1 of [C1]),

$$I_{f_1, f_2}(N, h) = \sum_a W(a) \mathcal{C}_{f_1, f_2}(a) + \mathcal{O}_\varepsilon(N^\varepsilon h^3).$$

Since the WIDTH is $\theta < 1/2$, we have the DIAGONAL terms, i.e. terms $\lll Nh$, arising from $a = 0$ here, i.e.

$$W(0) \mathcal{C}_{f_1, f_2}(0) = 2h \sum_{n \sim N} f_1(n) f_2(n) \lll Nh,$$

that are bigger than the TAILS; these will be, then, ignored. As regards terms $a \neq 0$ we apply our MAIN LEMMA, getting (recall W EVEN)

$$\sum_{a \neq 0} W(a) \mathcal{C}_{f_1, f_2}(a) = \sum_{a > 0} W(a) (\mathcal{C}_{f_1, f_2}(a) + \mathcal{C}_{f_1, f_2}(-a)) \sim \sum_{a \neq 0} W(a) N \sum_{\ell|a} \frac{1}{\ell} \sum_{(d, q)=1} \frac{g_1(\ell d)}{d} \frac{g_2(\ell q)}{q},$$

SAY, i.e., PLUS REMAINDERS, from Main Lemma and the trivial $W(a) \ll h$ (uniform $\forall a$), that are IN THE FINAL I_{f_1, f_2} ESTIMATE. Thus, we are left with the main term

$$N \sum_{\ell \leq 2h} \frac{1}{\ell} \sum_{b \neq 0} W(\ell b) \sum_{(d, q)=1} \frac{g_1(\ell d)}{d} \frac{g_2(\ell q)}{q} \lll N \sum_{\ell \leq 2h} \frac{1}{\ell} \left| \sum_{b \neq 0} W(\ell b) \right| \lll Nh,$$

i.e., DIAGONAL terms (into remainders above), from Lemma 4 [C1] (apply elementary calculations), entailing

$$\sum_{\substack{a \\ a \equiv 0(q)}} W(a) = 2q \left\| \frac{h}{q} \right\| \lll h, \quad \text{UNIFORMLY } \forall q \in \mathbb{N}.$$

We need, in order to complete Proposition 1, to treat, also, J_{f_1, f_2} . This is done in an analogous manner (as for I_{f_1, f_2}), but THIS TIME the (elementary) DISPERSION (method we are performing) HAS (“mixed”) MAIN TERMS to take into account, here (compare Lemma 2, [C1], & Selberg integral calculations on the Journal):

$$J_{f_1, f_2} = \int_N^{2N} \left(\sum_{x < n \leq x+h} f_1(n) - M_{f_1}(h) \right) \left(\sum_{x < m \leq x+h} f_2(m) - M_{f_2}(h) \right) dx = \int_N^{2N} \sum_{x < n, m \leq x+h} f_1(n) f_2(m) dx \\ - M_{f_1}(h) \int_N^{2N} \sum_{x < m \leq x+h} f_2(m) dx - M_{f_2}(h) \int_N^{2N} \sum_{x < n \leq x+h} f_1(n) dx + N M_{f_1}(h) M_{f_2}(h).$$

Here

$$\int_N^{2N} \sum_{x < n \leq x+h} f_1(n) dx = \sum_{N < n \leq 2N+h} f_1(n) \int_{\substack{N \leq x \leq 2N \\ n-h \leq x < n}} dx = \sum_{N+h < n < 2N-h} f_1(n) \int_{n-h}^n dx + \mathcal{O}_\varepsilon(N^\varepsilon h^2) = \\ = h \sum_{n \sim N} f_1(n) + \mathcal{O}_\varepsilon(N^\varepsilon h^2)$$

and in the same way

$$\int_N^{2N} \sum_{x < m \leq x+h} f_2(m) dx = h \sum_{m \sim N} f_2(m) + \mathcal{O}_\varepsilon(N^\varepsilon h^2).$$

Hence, from

$$\sum_{n \sim N} f_1(n) = \sum_d g_1(d) \left(\left[\frac{2N}{d} \right] - \left[\frac{N}{d} \right] \right) = N \sum_d \frac{g_1(d)}{d} + \mathcal{O}_\varepsilon(N^\varepsilon D)$$

and

$$\sum_{m \sim N} f_2(m) = N \sum_d \frac{g_2(d)}{d} + \mathcal{O}_\varepsilon(N^\varepsilon Q)$$

(recalling M_f definition and that $D \leq Q$, here) we have

$$M_{f_1}(h) \int_N^{2N} \sum_{x < m \leq x+h} f_2(m) dx + M_{f_2}(h) \int_N^{2N} \sum_{x < n \leq x+h} f_1(n) dx = 2N M_{f_1}(h) M_{f_2}(h) + \mathcal{O}_\varepsilon(N^\varepsilon (Q + h^3))$$

(where, again, TAILS are absorbed by DIAGONAL terms), whence we only need to rewrite the integral of our f_1, f_2 double sum; as this can be done as above (where there was an extra “sign”) in analogous manner to Lemma 2 [C1], we get, with $S(a) \stackrel{def}{=} \max(h - |a|, 0)$,

$$J_{f_1, f_2} = \sum_{a \neq 0} S(a) \mathcal{C}_{f_1, f_2}(a) - N M_{f_1}(h) M_{f_2}(h) + \mathcal{O}_\varepsilon(N^\varepsilon (Nh + Qh^2)).$$

A standard weighted (with, this time S instead of W , but, however, S EVEN) sum of $a \neq 0$ -correlations, applying MAIN LEMMA, gives feasible (i.e., already into above) remainders, together with main term

$$\sum_{a \neq 0} S(a) N \sum_{\ell | a} \frac{1}{\ell} \sum_{(d, q)=1} \sum \frac{g_1(\ell d)}{d} \frac{g_2(\ell q)}{q} = N \sum_{\ell \leq h} \frac{1}{\ell} \left(\sum_{b \neq 0} S(b) \right) \sum_{(d, q)=1} \sum \frac{g_1(\ell d)}{d} \frac{g_2(\ell q)}{q} = \\ = N \sum_{\ell \leq h} \frac{1}{\ell} \left(\frac{h^2}{\ell} + \mathcal{O}(h) \right) \sum_{(d, q)=1} \sum \frac{g_1(\ell d)}{d} \frac{g_2(\ell q)}{q} = Nh^2 \sum_{\ell \leq h} \frac{1}{\ell^2} \sum_{(d, q)=1} \sum \frac{g_1(\ell d)}{d} \frac{g_2(\ell q)}{q} + \mathcal{O}_\varepsilon(N^\varepsilon Nh),$$

(we'll leave, writing \sim , last DIAGONAL terms), where we use (compare [C1], section 3 on Journal, calculations)

$$\sum_{b \neq 0} S(\ell b) = 2 \sum_{b \leq \frac{h}{\ell}} (h - \ell b) = \frac{h^2}{\ell} + \mathcal{O}(h), \quad \forall \ell \leq h$$

to get main term

$$\begin{aligned} \sum_{a \neq 0} S(a) N \sum_{\ell|a} \frac{1}{\ell} \sum_{(d,q)=1} \frac{g_1(\ell d)}{d} \frac{g_2(\ell q)}{q} &\sim N h^2 \sum_{\ell \leq h} \frac{1}{\ell^2} \sum_{(d',q')=1} \frac{g_1(\ell d')}{d'} \frac{g_2(\ell q')}{q'} \\ &\sim N h^2 \sum_{\ell=1}^{\infty} \frac{1}{\ell^2} \sum_{(d',q')=1} \frac{g_1(\ell d')}{d'} \frac{g_2(\ell q')}{q'} \end{aligned}$$

(last \sim leaves, again, DIAGONAL terms, compare quoted [C1] section) and this is (change variables)

$$= N h^2 \sum_{\ell=1}^{\infty} \sum_{(d,q)=\ell} \frac{g_1(d)}{d} \frac{g_2(q)}{q} = N h^2 \sum_{d \leq D} \sum_{q \leq Q} \frac{g_1(d)}{d} \frac{g_2(q)}{q} = N M_{f_1}(h) M_{f_2}(h),$$

from M_f definition. Hence, main term cancels and we have

$$J_{f_1, f_2}(N, h) \ll N h + N^\delta (DQ)^{\frac{7}{8}} Q^{\frac{11}{48}} h^2 + N^{1-2\delta/3} h^2 + Q h^2. \quad \square$$

PROOF OF PROPOSITION 2. In the case $\lambda_1 + \lambda_2 < 1$, we have that $R_{D,Q}(|a|) \ll DQ$ suffices, in order to get both the non-trivial estimates for $J_{f_1, f_2}(N, h)$ and $I_{f_1, f_2}(N, h)$ (see, also, the above considerations).

Instead, when $\frac{1}{2} < \lambda_1 \leq \lambda_2$ and $42\lambda_1 + 53\lambda_2 < 48$, apply Proposition 1. \square

4. Proof of the Theorem.

The proof is a simple exercise from Proposition 1, setting $\lambda_1 = \lambda_2 = \lambda$, $D = Q$, $f_1 = f_2 = f = g * \mathbf{1}$. \square

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