ON THE SYMMETRY OF ARITHMETICAL FUNCTIONS IN ALMOST ALL SHORT INTERVALS

By

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Abstract. We study the mean-square (over $x \in [N, 2N]$) of the "symmetry sum" of arithmetical functions f, namely $\sum_{|n-x|\le h} \mathrm{sgn}(n-x) f(n)$; in this way we are checking the symmetry of f in "almost all" short intervals by elementary methods; actually, we use the Large Sieve alone, together with elementary considerations.

1. Introduction and statement of the results.

In this paper we study the symmetry of a particular class of arithmetical functions (generalizing the results of our previous papers) in almost all short intervals. As in the previous papers [2], [3] and [4] on the symmetry of (respectively) the function $\omega(n)$ (i.e. the prime-divisors function), $\Lambda(n)$ (i.e. the von Mangoldt function) and $d(n)$ (the number of divisors of n) we use here the Large Sieve to derive a similar bound for particular classes of arithmetical functions, which generalize each of the quoted works. In fact, we will take into account the arguments from all of these papers; one feature in common being the use of the Large Sieve (applied to the periodic function $\chi_q(x)$, see the following). In our paper [4] we use the "flipping" property of the divisor function $d(n)$ (i.e., we have the same contributes from divisors of n up to and, also, beyond $\sqrt{n}$); this allows a straightforward application of the Large Sieve (see the following Lemma 1), more in general, even to a particular class of arithmetical functions to which $d(n)$ belongs (see the following Theorem 3). As regards the function $\omega(n)$, in our paper [2] we have simply estimated the contributes up to (roughly) $\sqrt{N}$, applying the cancelation due to the sign, while we have estimated trivially (i.e., with absolute values instead of signs) the other moduli. We generalize, here, this estimate of "low" modules (which is non-trivial up to slightly more than $\sqrt{N}$), by our Theorem 1. However, when no flipping can be applied, we use an argument (our Lemma 3) which generates "sporadic" terms, to be estimated by a "fragmented Large-Sieve" (our Lemma 2); this doesn't apply to "extremely high" modules $d > N/2h$ (see the following). In order to avoid these divisors, we have two choices: we can exclude them, providing a result only for a particular class of functions (see our Theorem 2); or, also, if the Dirichlet convolution $f * \mathbf{1}$ is "enough symmetryc" (see also [3]), we can get a result on the "average-symmetry" of distribution of f (see our Corollary 1). In fact, our Corollary 1 (see section 4) provides a generalization of our work [3]; there, we use the property of the Λ function, of having a "very symmetryc" summatory function (over the divisors), i.e. $\Lambda * \mathbf{1} = \log$ (here $\mathbf{1}(n) = 1$, $\forall n \in \mathbf{N}$); in fact the log-function has a very small symmetry sum $S^{\pm}_{\log}$ (see the following), even for **all** x (see [3]). In both cases, we use the Large Sieve for the "low" divisors (they're not "small moduli", as they go slightly beyond $\sqrt{N}$), see Lemma 1, and the fragmented Large-Sieve, see Lemma 2, for the "high" divisors. Finally, see section 4, we also provide some examples of arithmetic functions for our three Theorems; also, we give some examples in which there are individual results, i.e. we get enough symmetry for **all** short intervals, rather than for almost all. Actually, this is possible, for example, for the square-free numbers (and, in the same way, for k-free numbers).

First of all, we consider n to be in a short interval of the kind $[x-h, x+h]$ (we define it short, as usual, if $h = o(x)$ and $h(x) \to \infty$ as $x \to \infty$) and we form the "symmetry sum" (here and in the sequel $x \in \mathbf{N}$ and $x \to \infty$):

$$S^{\pm}_f(x) := \sum_{x-h \le n \le x+h} f(n)\mathrm{sgn}(n-x)$$

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(here $\mathrm{sgn}(t) := t/|t|$, $\forall t \in \mathbf{R}-\{0\}$, $\mathrm{sgn}(0) = 0$); then we consider its "mean square" (with $h = h(N) \in \mathbf{N}$ independent of x, where $N \in \mathbf{N}$ and $N \to \infty$):

$$I_f(N,h) := \sum_{N<x\le 2N} \left| S_f^{\pm}(x) \right|^2 .$$

On the variables of summation we mean by $d \sim D$ that $D < d \le 2D$. In the sequel we'll write $L := \log N$. We now give our results.

We start from the generalization of our Theorem 1 of [2]:

Theorem 1. *Let N, h be natural numbers, with $h = h(N) \to \infty$ and $h = o(N)$ as $N \to \infty$; assume that $A = A(N,h) \ge 1$ is a real parameter. Then, defining $M_f := \max_{n \le A\sqrt{N}} |f(n)|$,*

$$\sum_{x\sim N} \left| \sum_{|n-x|\le h} \mathrm{sgn}(n-x) \sum_{\substack{d|n \\ d\le A\sqrt{N}}} f(d) \right|^2 \ll M_f^2 \, N A^2 h L^3 .$$

Remark. We might also consider sums on d with $f(d)g(\frac{n}{d})$ instead of $f(d)$, for a suitable $g \in \mathcal{S}$ (see before Theorem 3), but we will not, for clarity.

Actually, the sum over the divisors can be even wider (thanks to the "Fragmented-Large-Sieve", i.e. our Lemma 2), as stated in our

Theorem 2. *Let $N, h \in \mathbf{N}$, with $h = h(N) \to \infty$ and $h = o(N)$, as $N \to \infty$. Then, assuming $M_f := \max_{n \le \frac{N}{2h}} |f(n)|$, we have*

$$\sum_{x\sim N} \left| \sum_{|n-x|\le h} \mathrm{sgn}(n-x) \sum_{\substack{d|n \\ d\le \frac{N}{2h}}} f(d) \right|^2 \ll M_f^2 \, N^{4/3} h L^{13/3} .$$

(We emphasize that the same remark of Theorem 1 is still valid). We remark that, of course, both this Theorem and its Corollary 1 (see section 4) are non-trivial when $h \ge N^{1/3}L^B$, $B > 0$ suitable (see also [3]). We now come to the generalization of the results of [4]. For this reason, we will define a set of arithmetical functions, all of which are **quasi-constant** (see the following, in particular requirement 3), because a convolution of two such functions is **quasi-symmetryc** (i.e., the mean-square of its symmetry sum is enough small). The arithmetical functions f and g appearing in Theorem 3 will satisfy the following requirements:

1) f is a completely additive or completely multiplicative arithmetical function with complex values, whose domain is extended to positive rationals in the obvious way;
2) $M_f := \max_{\mathbf{Q}^+} |f| \ll N^c$, for a certain (absolute) constant $c > 0$;
3) $\left| f(m) - f\left(\frac{x}{d}\right) \right| \ll \left| m - \frac{x}{d} \right| h^{-C}$, for a fixed $C > 0$, uniformly in all the variables involved, i.e. $\forall x \in [N, 2N]$, $\forall d \le \sqrt{x}$, $\forall m \in \left[\frac{x-h}{d}, \frac{x+h}{d}\right]$.

(The last one gives enough smooth functions, with no regularity condition on the derivatives).

We will indicate the set of these functions with $\mathcal{S}$; also, as we suppose them to be completely additive or completely multiplicative we will write, for definition, $\mathcal{S}_{\mathcal{A}}$ for those f completely additive and $\mathcal{S}_{\mathcal{M}}$ for those f completely multiplicative. In order to give Theorem 3, we need to abbreviate the remainders, so we set (for $f, g \in \mathcal{S}$)

$$R_{f,g}(N,h) := (M_f^2 + M_g^2) N h^{2-2C} L^2 + M_f^2 M_g^2 (h^2/N + N),$$

where (for each arithmetical function f) $m_f := \min_{1<n\le\sqrt{2N}} |f(n)|$, $M_f := \max_{n\le\sqrt{2N}} |f(n)|$.

Then we have the following

Theorem 3. *Let $N, h \in \mathbf{N}$ with $h = h(N) \to \infty$ and $h = o(N)$ as $N \to \infty$ and assume that $f, g \in \mathcal{S}$. Then, abbreviating I for $I_{f*g}(N,h)$*

i) $f, g \in \mathcal{S}_{\mathcal{A}} \;\Rightarrow\; I \ll NhL^3 M_f^2 M_g^2 + R_{f,g}(N,h)$.

ii) $f \in \mathcal{S}_{\mathcal{A}}$, $g \in \mathcal{S}_{\mathcal{M}} \;\Rightarrow\; I \ll NhL^3 \left(\frac{M_f^2 M_g^2}{m_g^4} + M_f^2 M_g^4 \right) + R_{f,g}(N,h)$.

iii) $f, g \in \mathcal{S}_{\mathcal{M}} \;\Rightarrow\; I \ll NhL^3 \left(\frac{M_f^4 M_g^2}{m_g^4} + \frac{M_g^4 M_f^2}{m_f^4} \right) + R_{f,g}(N,h)$.

The paper is organized as follows:
-) in section 2 we give the Lemmas to prove our Theorems;
-) in section 3 we prove our Theorems;
-) in section 4 we give some examples.

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2. Lemmas.

Lemma 1 *Let Q and N be natural numbers, $M \in \mathbf{Z}$ and $\lambda_{a,q}$ be complex numbers ($\forall a, q \in \mathbf{N}$); then*

$$\sum_{n=M+1}^{M+N} \left| \sum_{q \le 2Q} \sum_{a \le q}{}^{*} \lambda_{a,q} e_q(an) \right|^2 = (N + \mathcal{O}(Q^2)) \sum_{q \le 2Q} \sum_{a \le q}{}^{*} |\lambda_{a,q}|^2 .$$

(Here, as usual, the $*$ in the sum means that $(a, q) = 1$). This is a version of the well-known Large Sieve inequality (see, for example, [1], or [5]). We remark that we can prove it also using Hilbert's inequality (see the proof of Lemma 3 of [4] and the relative considerations). In this way, actually, we get this version of the Large Sieve in [4] (see Lemma 3 of [4]) which gives an exact asymptotic estimate (see the consequences on the Theorems and Corollaries of [4]). However, we will confine, here, to the application of the "classical" version (with the only upper bound $\mathcal{O}(N + Q^2)$, instead of $N + \mathcal{O}(Q^2)$, see above) for the proof of our Theorems 1 and 2 (but we give the "asymptotic" version for completeness).

Lemma 2 *Let J, N be natural numbers, with $J = o(N)$ as $N \to \infty$ and f be an arithmetical function with $M_f := \max_{n \in \mathbf{N}} |f(n)|$. Then*

$$\chi_q(x) := \sum_{\substack{|n-x| \le h \\ n \equiv 0 (\bmod q)}} \quad \Rightarrow \quad \sum_{x \sim N} \left| \sum_{q \sim J} \chi_q(x) f\left(\left[\frac{x+h}{q} \right] \right) \right|^2 \ll M_f^2 N J h L^3 .$$

(Here the implied constant depends at most on J, N).

Proof. We first write, say

$$S_J(x) := \sum_{q \sim J} \chi_q(x) f\left(\left[\frac{x+h}{q} \right] \right),$$

whence the sum on q is (see the proof of Theorem 1):

$$S_J(x) = \sum_{d \le 2J} \sum_{n \sim \frac{J}{d}} \frac{1}{n} f\left(\left[\frac{x+h}{nd} \right] \right) \sum_{j \le d}{}^{*} c_{j,d} e_d(jx),$$

and then expand its square to get (see also [3])

$$\sum_{x \sim N} |S_J(x)|^2 = \sum_{n_1, n_2 \le 2J} \frac{1}{n_1} \frac{1}{n_2} \sum_{\substack{d_1 \sim J/n_1 \\ d_2 \sim J/n_2}} \sum_{j_1 \le d_1}{}^{*} \sum_{j_2 \le d_2}{}^{*} c_{j_1,d_1} \overline{c_{j_2,d_2}} \times$$

$$\times \sum_{x \sim N} f\left(\left[\frac{x+h}{n_1 d_1} \right] \right) \overline{f\left(\left[\frac{x+h}{n_2 d_2} \right] \right)} e\left(\left(\frac{j_1}{d_1} - \frac{j_2}{d_2} \right) x \right).$$

This last sum over x is estimated trivially, when $j_1/d_1 = j_2/d_2$, as $\mathcal{O}(N M_f^2)$; while can be bounded by means of Lemma 2 of [4] when j_1/d_1 and j_2/d_2 are distinct Farey fractions. In fact we can break the range $[N, 2N]$ into $\mathcal{O}(N/J)$ "fragmented" ranges (each of length at most $\mathcal{O}(J)$) in which the functions f are constant (being constant their arguments); then, we argue as in Lemma 2 of [4], with $k := j/d$ and $C_k := c_{j,d}$ (and we use the classical estimate for exponential sums, see [9]):

$$\sum_{x \sim N} |S_J(x)|^2 \ll M_f^2 \sum_{n_1, n_2 \le 2N} \frac{1}{n_1} \frac{1}{n_2} \sum_{k_r, k_s} |C_{k_r}| \, |C_{k_s}| \min\left(N, \frac{N}{J \, \|k_r - k_s\|} \right)$$

$$\ll M_f^2 \sum_{n_1,n_2\le 2N} \frac{1}{n_1 n_2}\left(N\sum_k |C_k|^2 + \frac{N}{J}\sum_{k_r\neq k_s} |C_{k_r}|\,|C_{k_s}|\,\frac{1}{\|k_r-k_s\|}\right),$$

which is, by Lemma 2 of [4], with $\delta \ge 1/J^2$ (the k are δ-well-spaced), at most

$$\ll M_f^2 \sum_{n_1,n_2\le 2N} \frac{1}{n_1 n_2} L\,(N+NJ)\sum_k |C_k|^2 \ll M_f^2\, NJhL^3,$$

since $\sum_k |C_k|^2 \ll h$; this concludes the proof of the Lemma.

Remark. We explicitly remark that the remainders of this Lemma are non-optimal, due to the non-optimality of the argument used; also, for particular functions f, we may try to estimate the exponential sums arising from the fragmented x-range, to get more cancelation.

Lemma 3. *Let N,h be natural numbers, with $h=h(N)\to\infty$ and $h=o(N)$ as $N\to\infty$, and assume that $A=A(N)\ge 1$ is a real parameter. Then, defining* $M_f := \max_{A\sqrt{N}<n\le \frac{2N+h}{2h}} |f(n)|$,

$$\sum_{x\sim N}\left|\sum_{A\sqrt{N}<d\le\frac{x+h}{2h}} f(d)\chi_d(x)\right|^2 \ll \left(\frac{N^{3/2}}{A}hL^5 + \frac{h^2}{A^2} + NL^3\right)M_f^2.$$

Proof. First of all, write $\Sigma_2(x) := \sum_{A\sqrt{N}<d\le\frac{x+h}{2h}} f(d)\chi_d(x)$ and use $\chi_d(x)$ def.(Lemma 2):

$$\Sigma_2(x) = \sum_{j\in\mathbf{N}}\left(\sum_{\substack{A\sqrt{N}<d\le\frac{x+h}{2h}\\ \frac{x}{j}<d<\frac{x+h}{j}}} f(d) - \sum_{\substack{A\sqrt{N}<d\le\frac{x+h}{2h}\\ \frac{x-h}{j}<d<\frac{x}{j}}} f(d)\right) + \mathcal{O}\left(\left(d(x-h)+d(x)+d(x+h)\right)M_f\right)$$

($d=\frac{x-h}{j}$, $d=\frac{x}{j}$ and $d=\frac{x+h}{j}$ give the divisor-functions-remainders; their mean-square is $\mathcal{O}(NM_f^2L^3)$, see [7], which will be omitted for the moment). This series is genuinely finite, due to the constraints on d; in fact,

$$\Sigma_2(x) = \sum_{\frac{2(x-h)h}{x+h}<j<\frac{x+h}{A\sqrt{N}}}\left(\sum_{\substack{A\sqrt{N}<d\le\frac{x+h}{2h}\\ \frac{x}{j}<d<\frac{x+h}{j}}} f(d) - \sum_{\substack{A\sqrt{N}<d\le\frac{x+h}{2h}\\ \frac{x-h}{j}<d<\frac{x}{j}}} f(d)\right) =$$

$$= \sum_{2h<j\le\frac{x-h}{A\sqrt{N}}}\ \sum_{|d-\frac{x}{j}|\le\frac{h}{j}} \mathrm{sgn}(d-x/j)f(d) + \mathcal{O}\left(M_f\left(\sum_{\frac{2(x-h)h}{x+h}<j\le 2h} + \sum_{\frac{x-h}{A\sqrt{N}}<j\le\frac{x+h}{A\sqrt{N}}}\right)\left(\frac{h}{j}+1\right)\right)$$

$$= \sum_{2h<j\le\frac{x-h}{A\sqrt{N}}}\ \sum_{|d-\frac{x}{j}|\le\frac{h}{j}} \mathrm{sgn}(d-x/j)f(d) + \mathcal{O}\left(\left(\frac{h^2}{N}+\frac{h}{A\sqrt{N}}+1\right)M_f\right),$$

since $h=o(N)$ implies $\dfrac{2(x-h)h}{x+h}=2h+o(h)$; then the inner sum on d is **sporadic**, i.e. has at most **one** term (and the bound $\mathcal{O}(1)$ is non-optimal). Henceforth, we have this sporadic term if and only if $](x-h)/j,(x+h)/j]$ contains an integer, which is unique and exactly $[\frac{x+h}{j}]$; the function $\chi_j(x)$ detects it, also giving (by the sign) its side w.r.t. $\frac{x}{j}$ (right or left of). Then (apart from the remainders, which we'll join later)

$$S(x) := \sum_{2h<j\le\frac{x-h}{A\sqrt{N}}}\ \sum_{\frac{x-h}{j}<d\le\frac{x+h}{j}} \mathrm{sgn}(d-x/j)f(d) = \sum_{2h<j\le\frac{x-h}{A\sqrt{N}}} \chi_j(x) f\left(\left[\frac{x+h}{j}\right]\right);$$

we argue as in [3], by a dyadic dissection of the j-range, to bound this sum by (and, also, to treat the following last sum)

$$L \max_{2h<J\le \frac{N-h}{A\sqrt{N}}} \sum_{j\sim J} \chi_j(x) f\left(\left[\frac{x+h}{j}\right]\right) + \sum_{\frac{N-h}{A\sqrt{N}}<j\le \frac{x-h}{A\sqrt{N}}} \chi_j(x) f\left(\left[\frac{x+h}{j}\right]\right).$$

We apply Lemma 2 to conclude.

3. Proof of the Theorems.

Proof of Theorem 1. Letting $\Sigma_1(x) := \sum_{d\le A\sqrt{N}} \chi_d(x) f(d)$ (see next line), we'll show

$$\sum_{x\sim N} |\Sigma_1(x)|^2 \ll NA^2 M_f^2\, hL^3.$$

Before the Large-Sieve, we first need to rearrange the finite Fourier expansion [4] of $\chi_d(x)$, defined in Lemma 2, as follows. We use $c_{at,bt} = \frac{1}{t} c_{a,b}$ to get

$$\chi_d(x) = \sum_{j<d} c_{j,d} e_d(jx) = \sum_{\substack{j<d\\(j,d)=t}} c_{j,d} e_d(jx) = \sum_{t|d} \frac{1}{t} \sum_{\substack{j'<(d/t)\\(j',(d/t))=1}} c_{j',d/t} e_{d/t}(j'x) = \sum_{t|d} \frac{t}{d} \sum_{\substack{r\le t\\(r,t)=1}} c_{r,t} e_t(rx)$$

whence

$$\Sigma_1(x) = \sum_{t\le A\sqrt{N}} \Big(\sum_{n\le \frac{A\sqrt{N}}{t}} f(tn)/n \Big) \sum_{j\le t}^{*} c_{j,t} e_t(jx)$$

($\sum^*$ is the summation over reduced classes, i.e. $(j,t)=1$). We can apply Lemma 1 to get

$$\sum_{x\sim N} \Big| \sum_{q\le A\sqrt{N}} \alpha_f(q) \sum_{j\le q}^{*} c_{j,q} e_q(jx) \Big|^2 \ll NA^2 \sum_{q\le A\sqrt{N}} |\alpha_f(q)|^2 \sum_{j<q} |c_{j,q}|^2,$$

where, say, (and by definition of $\chi_q(x)$) $\alpha_f(q) := \sum_{n\le \frac{A\sqrt{N}}{q}} \frac{f(qn)}{n}$, $\sum_{j<q} |c_{j,q}|^2 \ll \frac{h}{q}$; then we finally obtain the Theorem:

$$\sum_{x\sim N} \Big| \sum_{d\le A\sqrt{N}} f(d)\chi_d(x) \Big|^2 \ll NA^2 M_f^2\, hL^3.$$

Remark. We explicitly emphasize that (by Lemma 3 of [4]) the same result of this Theorem holds, with an additional logarithm (due to Lemma 3 [4]), whenever the summation over d is up to $\sqrt{x}$.

Proof of Theorem 2. We have

$$\sum_{|n-x|\le h} \mathrm{sgn}(n-x) \sum_{\substack{d|n\\ d\le \frac{N}{2h}}} f(d) = \sum_{d\le \frac{N}{2h}} f(d)\chi_d(x)$$

and (since Lemma 3 holds also when $A\sqrt{N} < d \le \frac{N}{2h}$, being the other summation terms $\frac{N}{2h} < d \le \frac{x+h}{2h}$ already contained in the remainders: see its proof) applying Theorem 1 and Lemma 3, together with the optimal choice $A = N^{1/6} L^{2/3}$, we get the Theorem.

Proof of Theorem 3. We start writing

$$S^{\pm}_{f*g}(x) = \sum_{|md-x|\le h} \Big(\sum_{d\le m} f(d) g(m) + \sum_{m\le d} g(m) f(d) \Big) \mathrm{sgn}(md-x) + \mathcal{O}\left(M_f M_g \left(\frac{h}{\sqrt{N}} + 1 \right) \right)$$

$$= \sum_{d\le\sqrt{x-h}} f(d) \sum_{\frac{x-h}{d}\le m\le \frac{x+h}{d}} g(m)\mathrm{sgn}\left(m - \frac{x}{d}\right) + \sum_{m\le\sqrt{x-h}} g(m) \sum_{\frac{x-h}{m}\le d\le \frac{x+h}{m}} f(d)\mathrm{sgn}\left(d - \frac{x}{m}\right)$$

$$+\mathcal{O}\left(M_f\, M_g \sum_{\sqrt{x-h}\le d\le\sqrt{x+h}} \left(\frac{h}{d}+1\right) + M_f\, M_g \left(\frac{h}{\sqrt{N}}+1\right) \right) =$$

$$= \sum_{d\le\sqrt{x}} f(d) g\left(\frac{x}{d}\right) \chi_d(x) + \sum_{m\le\sqrt{x}} g(m) f\left(\frac{x}{m}\right) \chi_m(x) + \mathcal{O}\left(\frac{M_f + M_g}{h^C} hL + M_f M_g \left(\frac{h^2}{N} + 1\right) \right),$$

the last equality coming from requirement 3) on f, g ($\in \mathcal{S}$, by hypothesis). (The "axiom" 2 is implicit, to avoid huge maxima M_f, M_g, while "axiom" 1 will be used now). Then we "separate the variables" n, d, x (depending on the hypotheses 1) on f, g) to get, by Theorem 1 (see the relative remark, after its proof) the three statements of the Theorem.

4. Examples.

First of all, we give the following consequence of Theorem 2, i.e. our

Corollary 1. *Let* $N, h \in \mathbf{N}$, *with* $h = h(N) \to \infty$ *and* $h = o(N)$, *as* $N \to \infty$. *Then*

$$\sum_{x \sim N} \Big| \sum_{j \le 2h} \sum_{|d - \frac{x}{j}| \le \frac{h}{j}} \mathrm{sgn}(d - \frac{x}{j}) f(d) \Big|^2 \ll M_f^2 N^{4/3} h L^{13/3} + \sum_{x \sim N} \Big| \sum_{|n-x| \le h} \mathrm{sgn}(n-x) \sum_{d|n} f(d) \Big|^2.$$

Proof. As for the proof of Theorem 2, by Lemma 1 and Lemma 3

$$\sum_{x \sim N} \Big| \sum_{|n-x| \le h} \mathrm{sgn}(n-x) \sum_{d|n, d \le \frac{x+h}{2h}} f(d) \Big|^2 \ll N^{4/3} h L^{13/3}.$$

Taking a glance at Lemma 3 proof, apart from negligible remainders for the mean-square (already contained in the ones of Lemma 3 and hence contained in that stated above), we get the Corollary:

$$\sum_{|n-x| \le h} \mathrm{sgn}(n-x) \sum_{\substack{d|n \\ d \le \frac{x+h}{2h}}} f(d) = \sum_{j \le 2h} \sum_{|d - \frac{x}{j}| \le \frac{h}{j}} \mathrm{sgn}(d - \frac{x}{j}) f(d) + \sum_{|n-x| \le h} \mathrm{sgn}(n-x) \sum_{d|n} f(d).$$

We can give, as an example for Theorem 1 (apart, of course, the case ω^*, see [2]), the case (see [6]) $f(n) := \log n$, and $A = 1$, i.e. $a(n) := \sum_{d|n, d \le \sqrt{n}} \log(d)$ has symmetry sum:

$$\sum_{|n-x| \le h} \mathrm{sgn}(n-x) a(n) \ll \sqrt{h} \log^{5/2} x, \qquad \text{a.a. } x \in [N, 2N]$$

("a.a." = "almost all", i.e. all $x \in [N, 2N]$ with at most $o(N)$ exceptions).

We can give, as an example for Theorem 2, the following function (see [6])

$$M_z(n) := \sum_{\substack{d|n \\ d<z}} \mu(d),$$

where we can choose any $z \le N/(2h)$. Also, we can give (apart Λ, seen in [3]), the same function as an instance of Corollary 1 (in fact, the second mean-square in the remainders is actually 0). Of course, the best instance of Theorem 3 is $f = g = \mathbf{1}$, i.e. the asymptotics for the symmetry of $d(n)$ in [4]. More in general, consider $f(d) = d^{-s}$, with $s = \sigma + it$, and $g = \mathbf{1}$; in this case we apply Theorem 1 (case 3, being both f, g completely multiplicative), provided $\sigma > -1$ (for the requirement 3 of the class $\mathcal{S}$), whatever is $t \in \mathbf{R}$ (but, of course, $s \in \mathbf{C}$ needs to be fixed). We will give asymptotics for this case in a future paper.

We come, at last, to an "individual" result: square-free numbers in **all** short intervals:

Theorem 4. *Let* $x, h \in \mathbf{N}$, *with* $h = h(x) \to \infty$ *and* $h = o(x)$, *as* $x \to \infty$. *Then*

$$\sum_{|n-x| \le h} \mathrm{sgn}(n-x) \mu^2(n) \ll h \log^{-A} x,$$

provided $h \ge \sqrt{N} \log^A x$. (Follows by writing $\mu^2(n) = \sum_{d^2|n} \mu(d)$, see [8].)

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